Homework No. 08B (2019 Spring)

PHYS 301: Theoretical Methods in Physics

Due date: None. Practice on spherical polar coordinates.

1. (**10 points.**) In spherical polar coordinates a point is coordinated by the intersection of family of spheres, cones, and half-planes, given by

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1a}$$

$$\theta = \tan^{-1} \sqrt{\frac{x^2 + y^2}{z^2}}, \tag{1b}$$

$$\phi = \tan^{-1} \frac{y}{x}, \tag{1c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2a}$$

$$\boldsymbol{\nabla} \theta = \hat{\boldsymbol{\theta}}\,\frac{1}{r}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2b}$$

$$\boldsymbol{\nabla} \phi = \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) Using the gradient operator in spherical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{3}$$

evaluate the following

$$\frac{\partial}{\partial r}\hat{\mathbf{r}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\theta}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\phi}} = 0, \tag{4a}$$

$$\frac{\partial}{\partial \theta}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}, \qquad \frac{\partial}{\partial \theta}\hat{\boldsymbol{\phi}} = 0, \tag{4b}$$

$$\frac{\partial}{\partial \phi}\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial \phi}\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}} = -(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}). \tag{4c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{r}}, \quad \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\phi}}, \tag{5a}$$

$$\boldsymbol{\nabla} \cdot (r^2 \hat{\mathbf{r}}), \quad \boldsymbol{\nabla} \cdot (r^2 \hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla} \cdot (r^2 \hat{\boldsymbol{\phi}}), \tag{5b}$$

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r} \right), \quad \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\theta}}}{r} \right), \quad \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{r} \right). \tag{5c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla} \times \hat{\mathbf{r}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\phi}}, \tag{6a}$$

$$\boldsymbol{\nabla} \times (r^2 \hat{\mathbf{r}}), \quad \boldsymbol{\nabla} \times (r^2 \hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla} \times (r^2 \hat{\boldsymbol{\phi}}), \tag{6b}$$

$$\boldsymbol{\nabla} \times \left(\frac{\hat{\mathbf{r}}}{r} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\theta}}}{r} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\phi}}}{r} \right). \tag{6c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.