Homework No. 08C (2019 Spring)

PHYS 301: Theoretical Methods in Physics

Due date: None. Practice on Dirac δ-function.

1. (**70 points.**) (Based on problem 1.44,45/1.43,44 Griffiths 4th/3rd edition.) Evaluate the following integrals:

   (a) $\int_2^6 dx\,[3x^2 - 2x - 3]\,\delta(x-3)$

   (b) $\int_{-7}^7 dx\,\sin x\,\delta(x-\pi)$

   (c) $\int_0^3 dx\,x^3\,\delta(x+1)$

   (d) $\int_{-2}^2 dx\,[3x+3]\,\delta(3x)$

   (e) $\int_{-2}^2 dx\,[3x+3]\,\delta(-3x)$

   (f) $\int_0^2 dx\,[3x+3]\,\delta(1-x)$

   (g) $\int_{-1}^1 dx\,9x^3\,\delta(3x+1)$

2. (**10 points.**) Evaluate the integral

$$\int_{-1}^{1} \frac{\delta(1-3x)}{x}\,dx. \tag{1}$$

   Hint: Be careful to avoid a possible error in sign.

3. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} dx\,\delta(1-2x)\Big[8x^2 + 2x - 1\Big]. \tag{2}$$

   (Caution: Be careful to avoid a possible error in sign.)

4. (**20 points.**) The Heaviside step function, named after Oliver Heaviside (1850-1925), has the integral representation

$$\theta(x) = \int_{-\infty}^{x} dx'\,\delta(x'). \tag{3}$$

   (a) Evaluate $\theta(x)$ for $x < 0$.

   (b) Evaluate $\theta(x)$ for $x > 0$.

(c) What about $\theta(0)$? We could postulate that

$$\theta(0) = \frac{1}{2}\left[\lim_{\varepsilon\to 0}\theta(x-\varepsilon) + \lim_{\varepsilon\to 0}\theta(x+\varepsilon)\right]. \tag{4}$$

Evaluate $\theta(0)$ obtained using Eq. (4).

(d) Plot $\theta(x)$ versus x.

5. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \tag{5}$$

Show that

$$\delta(x)\begin{cases}\to\infty, & \text{if} \quad x=0,\\ \to 0, & \text{if} \quad x\neq 0.\end{cases} \tag{6}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\,\delta(x) = 1. \tag{7}$$

Plot $\delta(x)$ for $\varepsilon\neq 0$ and identify ε in the plot.

6. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{x^2}{2\sigma}}. \tag{8}$$

Show that

$$\delta(x)\begin{cases}\to\infty, & \text{if} \quad x=0,\\ \to 0, & \text{if} \quad x\neq 0.\end{cases} \tag{9}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\,\delta(x) = 1. \tag{10}$$

Plot $\delta(x)$ for $\varepsilon\neq 0$ and identify ε in the plot.

7. (**30 points.**) (Based on problem 1.47/1.46 Griffiths 4th/3rd edition.)

(a) Express the charge density $\rho(\mathbf{r})$ of a point charge Q positioned at $\mathbf{r}_a$ in terms of δ-functions. Verify that the volume integral of ρ equals Q.

(b) Express the charge density of an infinitely long wire, of uniform charge per unit length λ and parallel to z-axis, in terms of δ-functions.

(c) Express the charge density of an infinite plate, of uniform charge per unit area σ and parallel to xy-plane, in terms of δ-functions.