Homework No. 12 (2019 Spring)

PHYS 301: Theoretical Methods in Physics

Due date: Monday, 2019 Apr 29, 9:00 AM, in class

1. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
   ```
   Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x],
   LegendreP[4,x] },{x,-1,1}]
   ```
   Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

2. (**20 points.**) Legendre polynomials are conveniently generated using the relation

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{1}$$

   where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

3. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation

$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{2}$$

   Verify this explicitly for $l = 0, 1, 2, 3, 4$.

4. (**20 points.**) Legendre polynomials satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{3}$$

   Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{4}$$

   Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$.

6. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x'). \tag{5}$$

This is for your information. No work is needed.