Homework No. 13 in lieu of the Final Exam (2019 Spring)

PHYS 301: Theoretical Methods in Physics

Due date: Monday, 2019 May 6, 10:00 AM

Vibrations of a (guitar) string of length a are described by the height of oscillation

$$h = h(x,t) \tag{1}$$

that satisfies the differential equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} \tag{2}$$

with boundary conditions

$$h(0,t) = 0, \tag{3a}$$

$$h(a,t) = 0, \tag{3b}$$

and initial conditions

$$h(x,0) = h_0(x), \tag{4a}$$

$$\left\{\frac{\partial}{\partial t}h(x,t)\right\}_{t=0} = 0. \tag{4b}$$

Here v is the speed of propagation given in terms of the tension T in the string (presumed to be uniform) and mass per unit length λ of the string, $v = \sqrt{T/\lambda}$. The given function $h_0(x)$ characterizes how the string is released initially.

1. Let $F(x)$ and $T(t)$ be eigenfunctions in terms of which the solution $h(x,t)$ can be described. Thus, the product

   $$F(x)T(t) \tag{5}$$

   satisfies the differential equation for $h(x,t)$. Substitute in Eq. (2) and rearrange to obtain

   $$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{T(t)}\frac{1}{v^2}\frac{\partial^2 T(t)}{\partial t^2}. \tag{6}$$

2. The left hand side of Eq. (6) is only dependent on x and the right hand side is only dependent on t. Argue that this can be satisfied for arbitrary x and t only if each side is equal to the same constant, say α. Note that α could be complex. This is called separation of variables. Thus, we have

   $$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \alpha = \frac{1}{T(t)}\frac{1}{v^2}\frac{\partial^2 T(t)}{\partial t^2}. \tag{7}$$

3. Rewrite the equation of $X(x)$ in the form

$$\frac{\partial^2 X}{\partial x^2} = \alpha X. \tag{8}$$

Verify that it permits the solution

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x}. \tag{9}$$

Show that the boundary conditions in Eq. (3) impose the conditions

$$A + B = 0, \tag{10a}$$

$$Ae^{\sqrt{\alpha}L} + Be^{-\sqrt{\alpha}L} = 0. \tag{10b}$$

Verify that $A = 0$ and $B = 0$ is a solution. However, it is a trivial solution, because it corresponds to no motion. Argue that Eq. (10) is also satisfied if

$$\det \begin{pmatrix} 1 & 1 \\ e^{\sqrt{\alpha}a} & e^{-\sqrt{\alpha}a} \end{pmatrix} = 0. \tag{11}$$

Thus, derive

$$\alpha = -m^2 \frac{\pi^2}{a^2}, \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{12}$$

Thus, conclude that $X(x)$ satisfies solutions of the form

$$X(x) = Ae^{im\pi\frac{x}{a}} + Be^{-im\pi\frac{x}{a}}. \tag{13}$$

Requiring this solution to satisfy the boundary conditions show that

$$X(x) = A' \sin\left(m\pi \frac{x}{a}\right), \tag{14}$$

where $A' = 2iA$. Observe that the boundary conditions do not determine A', it is left arbitrary.

4. Use the Wronskian to show that the eigenfunctions

$$\sin\left(m\pi\frac{x}{a}\right), \qquad m = 1, 2, 3, \ldots, \tag{15}$$

constitute linearly independent solutions. Verify that these functions satisfy the orthogonality relations

$$\frac{2}{a}\int_0^a dx \, \sin\left(m\pi\frac{x}{a}\right) \sin\left(m'\pi\frac{x}{a}\right) = \delta_{mm'}. \tag{16}$$

These functions also satisfy the completeness relation

$$\frac{2}{a}\sum_{m=1}^{\infty} dx \, \sin\left(m\pi\frac{x}{a}\right) \sin\left(m\pi\frac{x'}{a}\right) = \delta(x - x'), \tag{17}$$

which need not be proved here. This allows us to expand the desired solution $h(x,t)$ in terms of these eigenfunctions as

$$h(x,t) = \sum_{m=1}^{\infty} T_m(t) \sin\left(m\pi \frac{x}{a}\right), \tag{18}$$

where $T_m(t)$ are the respective components. Verify that $h(x,t)$ satisfies the boundary conditions.

5. Substituting this in the original differential equation show that

$$\sum_{m=1}^{\infty} \sin\left(m\pi \frac{x}{a}\right) \left[\frac{\partial^2 T_m}{\partial t^2} + \left(m\pi \frac{v}{a}\right)^2 T_m\right] = 0. \tag{19}$$

Using the completeness relation deduce the differential equations

$$\frac{\partial^2 T_m}{\partial t^2} = -\left(m\pi \frac{v}{a}\right)^2 T_m, \tag{20}$$

for each m. The solutions for these equations are of the form

$$T_m(t) = C_m \sin\left(m\pi \frac{v}{a} t\right) + D_m \cos\left(m\pi \frac{v}{a} t\right). \tag{21}$$

Thus, show that

$$h(x,t) = \sum_{m=1}^{\infty} \left[C_m \sin\left(m\pi \frac{v}{a} t\right) + D_m \cos\left(m\pi \frac{v}{a} t\right)\right] \sin\left(m\pi \frac{x}{a}\right). \tag{22}$$

Using the initial conditions show that

$$h_0(x) = \sum_{m=1}^{\infty} D_m \sin\left(m\pi \frac{x}{a}\right), \tag{23a}$$

$$0 = \sum_{m=1}^{\infty} C_m \left(m\pi \frac{v}{a}\right) \sin\left(m\pi \frac{x}{a}\right). \tag{23b}$$

Thus, learn that

$$C_m = 0. \tag{24}$$

Using orthogonality relations invert Eq. (23a) to derive

$$D_m = \frac{2}{a} \int_0^a dx\, h_0(x) \sin\left(m\pi \frac{x}{a}\right). \tag{25}$$

6. Together, summarize the solution to be

$$h(x,t) = \sum_{m=1}^{\infty} D_m \cos\left(m\pi \frac{v}{a} t\right) \sin\left(m\pi \frac{x}{a}\right), \tag{26}$$

where D_m is determined using the initial condition $h_0(x)$ using

$$D_m = \frac{2}{a} \int_0^a dx\, h_0(x) \sin\left(m\pi \frac{x}{a}\right) . \tag{27}$$

Find all D_m's for

$$h_0(x) = H \sin\left(\pi \frac{x}{a}\right) . \tag{28}$$

Hint: Use the orthogonality relations.