Midterm Exam No. 03 (Spring 2019)

PHYS 420: Electricity and Magnetism II

Date: 2019 Apr 10

1. (**20 points.**) A monochromatic plane electromagnetic wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{1a}$$

$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{1b}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constants. Assume no charges or currents. Using Maxwell's equations we can show that (need not be reproduced here)

$$\mathbf{k}\cdot\mathbf{E} = 0, \tag{2a}$$

$$\mathbf{k}\cdot\mathbf{B} = 0, \tag{2b}$$

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \tag{2c}$$

$$\mathbf{k}\times\mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}, \tag{2d}$$

where $\varepsilon_0\mu_0 = 1/c^2$. Evaluate the electromagnetic energy density

$$U = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H} \tag{3}$$

and the electromagnetic momentum density

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}. \tag{4}$$

Then, determine the ratio U/G.

2. (**20 points.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$, $\varepsilon_0\mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{5}$$

Th electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}, \qquad \mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{6}$$

Show that

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = nU, \tag{7}$$

where n is a number. Find n.

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,

$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{8}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyze what is obtained if you add a speed to an infinitely large superluminal speed, that is, $v_b \to \infty$. Hint: Inversion.

4. (**20 points.**) Prove that if u_α is a time-like vector and $u^\alpha a_\alpha = 0$ then a^α is necessarily space-like.
Hint: Evaluate these invariant relations in convenient frames.

5. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_2(t) = \sqrt{c^2 t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9}$$

This is the motion of a particle that comes to existence at $z_2 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_2(0) = 0$ and $y_2(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Another particle is at rest at z_1

$$z_1(t) = z_1, \tag{10}$$

such that $0 < z_1 < z_0$. Assume that both particles emit photons continuously.

(a) At what time do photons emitted by 2 first reach 1? Where is particle 2 when this happens?

(b) At what time is the last photon that reaches 2 emitted by 1? Where is particle 2 when this happens?

(c) Do all the photons emitted by 1 reach 2?

(d) Do all the photons emitted by 2 reach 1?