(Take Home) Final Exam (Spring 2019)

PHYS 420: Electricity and Magnetism II

Due date: Friday, 2019 May 10, 4:45 PM

1. (**100 points.**) The magnetic field associated to radiation fields is given by

$$c\mathbf{B}(\mathbf{r},t) = -\hat{\mathbf{r}} \times \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \left\{ \frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t') \right\}_{t'=t_r}, \tag{1}$$

where the contribution to the field comes at the retarded time

$$t_r = t - \frac{r}{c} + \hat{\mathbf{r}} \cdot \frac{\mathbf{r'}}{c}. \tag{2}$$

The associated electric field is given by

$$\mathbf{E}(\mathbf{r},t) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r},t), \tag{3}$$

and satisfies

$$c\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r},t). \tag{4}$$

Starting from the statement of conservation of electromagnetic energy density

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} = 0, \tag{5}$$

where the electromagnetic energy density

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}, \tag{6}$$

the flux of electromagnetic energy density (the Poynting vector)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{7}$$

$\mathbf{B} = \mu_0\mathbf{H}$; integrating over an infinitely large sphere centered about the sources; using divergence theorem to rewrite the second term; presuming the sources to be zero in the radiation zone; we deduce the power dP radiated into the solid angle $d\Omega$ to be

$$dP = \lim_{r\to\infty} r^2 d\Omega\, \hat{\mathbf{r}} \cdot \mathbf{S}. \tag{8}$$

(a) Using $\hat{\mathbf{r}} \cdot \mathbf{S} = \hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}) = (\hat{\mathbf{r}} \times (\mathbf{E}) \cdot \mathbf{H})$ show that this leads to the expression

$$\frac{\partial P}{\partial \Omega} = \lim_{r \to \infty} \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi} \right) \left| \frac{\mathbf{B}(\mathbf{r}, t)}{\frac{\mu_0}{4\pi} \frac{1}{r}} \right|^2. \tag{9}$$

Verify that $B / \left(\frac{\mu_0}{4\pi} \frac{1}{r} \right)$ has the dimensions of current. Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{10}$$

has the dimensions of resistance. Quantum phenomena in electromagnetism is characterized by the Planck's constant h and the associated fine-structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{\hbar c}, \tag{11}$$

a dimensionless physical constant. Verify that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} = \alpha \frac{\hbar}{e^2} = 29.9792458\,\Omega. \tag{12}$$

(b) A simple antenna consists of an infinitely thin conductor of length L carrying a time-dependent current. Let the conductor be centered at the origin and placed on the z axis such that

$$\mathbf{J}(\mathbf{r}', t') = \hat{\mathbf{z}}\, I_0 \sin \omega_0 t\, \delta(x') \delta(y') \theta(-L < 2z' < L). \tag{13}$$

The function θ equals 1 when the argument is a true statement, and zero otherwise. Show that

$$\int d^3 r' \left\{ \frac{\partial}{\partial t'} \mathbf{J}(\mathbf{r}', t') \right\}_{t' = t_r} = \hat{\mathbf{z}}\, \omega_0 I_0 \cos \left(\omega_0 t - 2\pi \frac{r}{\lambda_0} \right) \frac{\sin \left(\pi \frac{L}{\lambda_0} \cos \theta \right)}{\frac{\pi}{\lambda_0} \cos \theta}, \tag{14}$$

where $\omega_0 / c = 2\pi / \lambda_0$. Then, evaluate the expression for the magnetic field.

(c) Using Eq. (9) Show that

$$\frac{\partial P}{\partial \Omega} = P_0 \frac{\sin^2 \theta}{\pi} \cos^2 \left(\omega_0 t - 2\pi \frac{r}{\lambda_0} \right) \frac{\sin^2 \left(\pi \frac{L}{\lambda_0} \cos \theta \right)}{\cos^2 \theta}, \tag{15}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi} \right) I_0^2. \tag{16}$$

Evaluate the average power radiated into a solid angle using

$$\left\langle \frac{\partial P}{\partial \Omega} \right\rangle = \frac{1}{T} \int_0^T dt\, \frac{\partial P}{\partial \Omega}. \tag{17}$$

Show that

$$\left\langle \frac{\partial P}{\partial \Omega} \right\rangle = P_0 \frac{\sin^2\theta}{2\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0} \cos\theta\right)}{\cos^2\theta}. \tag{18}$$

Hint: Use the integral

$$\frac{1}{T} \int_0^T dt \, \cos^2(\omega_0 t + \delta) = \frac{1}{2}. \tag{19}$$

(d) Plot

$$g(\theta) = \sin^2\theta \frac{\sin^2\left(\pi \frac{L}{\lambda_0} \cos\theta\right)}{\cos^2\theta} \tag{20}$$

as a function of θ for $L = 0.1\lambda, 0.5\lambda, 1.0\lambda, 2.0\lambda, 3.0\lambda, 5.0\lambda$. Thus, discuss the angular distribution of the radiated power. Note that the radiated power is zero when

$$\theta = \cos^{-1}\left(n \frac{\lambda_0}{L}\right), \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{21}$$

Thus, the radiation pattern has a single lobe for $L < \lambda_0$. For $L > \lambda_0$ the radiation pattern exhibits a primary lobe bounded by $n = \pm 1$ and secondary lobes on either side of the primary lobe. Determine the number of lobes for $L = 3\lambda_0$. Using the area under $g(\theta)$ in your plot for $L = 3\lambda_0$ qualitatively estimate the percentage of power radiated into the primary lobe.