Homework No. 05 (2019 Spring)

PHYS 420: Electricity and Magnetism II

Due date: Wednesday, 2019 Feb 27, 2:00 PM, in class

0. (**0 points.**) Keywords for finding resource materials: Complete elliptic integrals; Magnetic vector potential and magnetic field for a circular loop carrying a steady current.

   Complete elliptic integrals of the first and second kind can be defined using the integral representations,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2\psi}}, \tag{1a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2\psi}, \tag{1b}$$

   respectively.

1. (**20 points.**) Verify that

$$K(0) = \frac{\pi}{2}, \tag{2a}$$

$$E(0) = \frac{\pi}{2}. \tag{2b}$$

   Then, verify that

$$E(1) = 1. \tag{3}$$

   Note that

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\cos\psi} \tag{4}$$

   is devergent. To see the nature of this divergence we introduce a cutoff parameter $\delta > 0$ and write

$$K(1) = \int_0^{\frac{\pi}{2}-\delta} \frac{d\psi}{\cos\psi}. \tag{5}$$

   Evaluate the integral, (using the identity $d(\sec\psi + \tan\psi)/d\psi = \sec\psi(\sec\psi + \tan\psi)$,) and show that

$$K(1) \sim \ln 2 - \ln\delta - \frac{\delta^2}{12} + \mathcal{O}(\delta)^4 \tag{6}$$

   has logarithmic divergence. Using Mathematica (or another graphing tool) plot $K(k)$ and $E(k)$ as functions of k for $0 \leq k < 1$.

2. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 k^{2n} = \frac{\pi}{2} \left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots \right], \tag{7a}$$

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 \frac{k^{2n}}{(1-2n)} = \frac{\pi}{2} \left[1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \ldots \right]. \tag{7b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the next-to-leading order contributions in the above series expansions by expanding the radical in Eqs.(1) as a series. Use

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \ldots, \tag{8a}$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \ldots. \tag{8b}$$

3. (**0 points.** Use as lecture notes.) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I \delta(\rho - a)\delta(z), \tag{9}$$

where the loop is chosen to be in the x-y plane with the origin as its center.

(a) Verify that

$$\int_S d\mathbf{a} \cdot \mathbf{j} = I, \tag{10}$$

where surface S is a half-plane of constant ϕ.

(b) Show that magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\hat{\boldsymbol{\phi}}'}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')}}. \tag{11}$$

(c) Substitute $\phi' - \phi \to \phi'$ and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos\phi'}}. \tag{12}$$

(d) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2\psi}}, \tag{13a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2\psi}, \tag{13b}$$

respectivly, the magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{4a}{\sqrt{z^2+(\rho+a)^2}}\left[\frac{2}{k^2}\Big\{K(k)-E(k)\Big\}-K(k)\right], \tag{14}$$

where

$$k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}. \tag{15}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (12) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a\cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals.

4. (**20 points.**) We have earlier found the magnetic vector potential to be zero everywhere along the symmetry axis of the circular loop. With our exact expression let us calculate an approximate expression for the magnetic vector potential very close to the axis. Using the power series expansions for the complete elliptic integrals show that

$$\frac{2}{k^2}\Big\{K(k)-E(k)\Big\}-K(k) = \frac{\pi}{16}k^2+\ldots. \tag{16}$$

Drop the next-to-leading order terms, valid when $k \ll 1$, and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{a^2\pi\rho}{[z^2+(\rho+a)^2]^{\frac{3}{2}}}. \tag{17}$$

Check that $\mathbf{A}=0$ on the axis.

5. (**0 points.** Use as lecture notes.) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I\delta(\rho-a)\delta(z), \tag{18}$$

where the loop is chosen to be in the x-y plane with the origin as its center.

   (a) Using Bio-Savart law and completing the integrals involving δ-functions show that magnetic field has the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \frac{\Big[a^2\hat{\mathbf{z}}+az\hat{\boldsymbol{\rho}}'-a\rho(\hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{\phi}}')\Big]}{[z^2+\rho^2+a^2-2\rho a\cos(\phi-\phi')]^{\frac{3}{2}}}. \tag{19}$$

   (b) Substitute $\phi'-\phi\to\phi'$ and show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \frac{\Big[(a^2-a\rho\cos\phi')\hat{\mathbf{z}}+az\hat{\boldsymbol{\rho}}\cos\phi'+az\hat{\boldsymbol{\phi}}\sin\phi'\Big]}{[z^2+\rho^2+a^2-2\rho a\cos\phi']^{\frac{3}{2}}}. \tag{20}$$

(c) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1-k^2\sin^2\psi}}, \tag{21a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1-k^2\sin^2\psi}, \tag{21b}$$

respectivly, the magnetic field is

$$\begin{aligned}\mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\left[K(k) - \frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2}E(k)\right] \\ -\hat{\boldsymbol{\rho}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\frac{z}{\rho}\left[K(k) - \frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2}E(k)\right],\end{aligned} \tag{22}$$

where

$$k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}. \tag{23}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (12) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a\cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals. It is useful to identify

$$\int_0^{\frac{\pi}{2}} d\psi \frac{1}{(1-k^2\sin^2\psi)^{\frac{3}{2}}} = \frac{E(k)}{(1-k^2)}. \tag{24}$$