Homework No. 09 (2019 Spring)

PHYS 420: Electricity and Magnetism II

Due date: Wednesday, 2019 Apr 3, 2:00 PM, in class

0. (**0 points.**) Keywords for finding resource materials: Relativistic kinematics.

1. (**100 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,
$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{1}$$
Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector
$$x^\alpha = (ct, \mathbf{x}). \tag{2}$$
   (a) Velocity: The four-vector associated with velocity is constructed as
   $$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{3}$$
      i. Using Eq. (1) deduce
      $$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{4}$$
      Then, show that
      $$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{5}$$
      Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.
      ii. Further, show that
      $$u^\alpha u_\alpha = -c^2. \tag{6}$$
      Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?
      iii. Write down the form of the velocity four-vector in the rest frame of the particle?
   (b) Momentum: Define momentum four-vector in terms of the mass m of the particle as
   $$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{7}$$
   Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)
   $$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{8}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{9a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{10}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{11}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{12}$$

i. Show that

$$a^\alpha = \gamma\left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a}\right), \tag{13}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{14}$$

is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (6) and taking derivative with respect to proper time show that

$$u^\alpha a_\alpha = 0. \tag{15}$$

Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (15) derive the identity

$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\gamma^3. \tag{16}$$

iv. Show that

$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4\right) \tag{17}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where

$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{18}$$

is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration

$$a^\alpha a_\alpha = \mathbf{a}_0 \cdot \mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)^2 \gamma^2\right]\gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c}\right)^2\right]\gamma^6. \tag{19}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{20}$$

And, in a particular frame, if $\mathbf{v}\perp\mathbf{a}$ (corresponding to circular motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{21}$$

(d) Force: The force four-vector is defined as

$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma\right), \tag{22}$$

where the force $\mathbf{F}$, identified (or defined) as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{23}$$

is the force in Newtonian physics. Starting from Eq. (10) derive the relation

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{24}$$

which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = m a^\alpha. \tag{25}$$

Show that these involve the relations

$$\mathbf{F} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{26a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{26b}$$

2. (**20 points.**) Lorentz transformation relates the energy of momentum of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{27}$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{28}$$