Homework No. 11 (2019 Spring)

PHYS 420: Electricity and Magnetism II

Due date: Wednesday, 2019 Apr 24, 2:00 PM, in class

0. (**0 points.**) Keywords for finding resource materials: Electrodynamics of moving bodies; Retarded time; Delta functions.

1. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{1}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate

$$\delta(ax^2 + bx + c). \tag{2}$$

2. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{3}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate

$$\delta(x^3 - 6x^2 + 11x - 6). \tag{4}$$

3. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\big[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}\big]}{(x^2 + y^2)} \frac{(x^2 + y^2)(1 - \beta^2)}{\big[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2\big]^{\frac{3}{2}}}, \tag{5a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t). \tag{5b}$$

(a) Consider the distribution

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2} \frac{\epsilon}{(x^2 + \epsilon)^{\frac{3}{2}}}. \tag{6}$$

Show that

$$\delta(x) \begin{cases} \to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{7}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{8}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\rho}}}{\rho} \delta(z - ct), \tag{9a}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct) = 2cq \frac{\mu_0}{4\pi} \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct), \tag{9b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2 + y^2}$. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this field configuration using a diagram.