Midterm Exam No. 02 (Spring 2019)

PHYS 510: Classical Mechanics

Date: 2019 Mar 28

1. (**20 points.**) Consider a wheel rolling on a horizontal surface. The following distinct

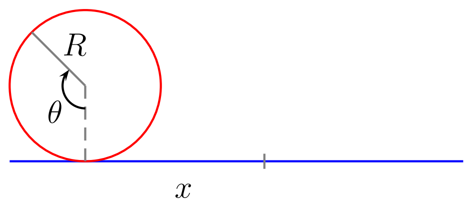

Figure 1: Problem 1.

types of motion are possible for the wheel:

$$\begin{aligned} x &< \theta R, \qquad \text{slipping (e.g. in snow)}, \\ x &= \theta R, \qquad \text{perfect rolling}, \\ x &> \theta R, \qquad \text{sliding (e.g. on ice)}. \end{aligned} \tag{1}$$

Differentiation of the these relations leads to the characterizations, $v < \omega R$, $v = \omega R$, and $v > \omega R$, respectively, where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Assuming the wheel is perfectly rolling, at a given instant of time, the tendency of motion could be to slip, to keep on perfectly rolling, or to slide.

Deduce that while perfectly rolling the relative motion of the point on the wheel that is in contact with the surface with respect to the surface is exactly zero. Thus, conclude that the force of friction on the wheel is zero. The analogy is a mass at rest on a horizontal surface. However, while perfectly rolling, it is possible to have the tendency to slip or slide without actually slipping of sliding. The analogy is that of a mass at rest under the action of an external force and the force of friction. In these cases the force of friction is that of static friction and it acts in the forward or backward direction.

In the following we differentiate between the following:

(a) Tendency of the wheel is to slip (without actually slipping) while perfectly rolling.

(b) Tendency of the wheel is to keep on perfectly rolling.

(c) Tendency of the wheel is to slide (without actually sliding) while perfectly rolling.

Deduce the direction of the force of friction in the above cases. Determine if the friction is working against linear acceleration or angular acceleration.

Perfect rolling involves the contraint $x = \theta R$. Thus, using the D'Alembert principle and the idea of Lagrange multiplier we can write the Lagragian for a perfectly rolling wheel on a horizontal surface to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - F_s(x - \theta R), \tag{2}$$

where m is the mass of the wheel, I is the moment of inertia of the wheel, and F_s is the Lagrangian multiplier. Using D'Alembert's principle give an interpretation for the Lagrangian multiplier F_s. What is the dimension of F_s? Infer the sign of F_s for the cases when the tendency of the wheel is to slip or slide while perfectly rolling.

2. (**20 points.**) A system, characterized by the parameters ω, α, and β, and the dynamical parameter θ, is described by the equation of motion

$$\ddot{\theta} + \omega^2 \sin\theta + \alpha \ddot{\theta} \cos\theta + \beta \dot{\theta}^2 \sin\theta = 0. \tag{3}$$

Write the above equation of motion in the small angle approximation, to the leading order in θ.

3. (**20 points.**) Consider infinitesimal rigid translation in time, described by

$$\delta \mathbf{r} = 0, \quad \delta \mathbf{p} = 0, \quad \delta t = \delta \epsilon, \tag{4}$$

where $\delta\epsilon$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \tag{5}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \epsilon} = \int_{t_1}^{t_2} dt \frac{dH}{dt} = H(t_2) - H(t_1). \tag{6}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \tag{7}$$

That is, when the Hamiltonian is independent of time.

(d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \tag{8}$$

when the action has translation symmetry.

4. (**20 points.**) For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \tag{9a}$$
$$B = B(\mathbf{x}, \mathbf{p}, t), \tag{9b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{10}$$

Evaluate

$$\big[A, \big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[B, \big[C, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[C, \big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{11}$$

The result is called the Jacobi's identity.

5. (**20 points.**) (Refer Sec. 21, Dirac's book on Quantum Mechanics.)
For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \tag{12a}$$
$$B = B(\mathbf{x}, \mathbf{p}, t), \tag{12b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{13}$$

The product rule for Poisson braket can be stated in the following different forms:

$$\big[A_1A_2, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A_1\big[A_2, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A_1, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} A_2, \tag{14a}$$
$$\big[A, B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = B_1\big[A, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B_2. \tag{14b}$$

(a) Thus, evaluate, in two different ways,

$$\begin{aligned}\big[A_1A_2, B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= A_1B_1\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + A_1\big[A_2, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B_2 \\ &\quad + B_1\big[A_1, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} A_2 + \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B_2A_2, \end{aligned} \tag{15a}$$

$$\begin{aligned}\big[A_1A_2, B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= B_1A_1\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + B_1\big[A_1, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} A_2 \\ &\quad + A_1\big[A_2, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B_2 + \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} A_2B_2. \end{aligned} \tag{15b}$$

(b) Subtracting these results, obtain

$$(A_1B_1 - B_1A_1)\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} (A_2B_2 - B_2A_2). \tag{16}$$

Thus, using the definition of the commutation relation,

$$\big[A, B\big] \equiv AB - BA, \tag{17}$$

obtain the relation

$$\big[A_1, B_1\big]\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \big[A_2, B_2\big]. \tag{18}$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2, conclude that

$$\big[A_1, B_1\big] = i\hbar\big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{19a}$$

$$\big[A_2, B_2\big] = i\hbar\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{19b}$$

where $i\hbar$ is necessarily a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. If A's and B's are numbers, then, because their commutation relation is equal to zero, we necessairily have $\hbar = 0$. But, if the commutation relation of A's and B's is not zero, then finite values of $\hbar$ is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant $\hbar$ is a real number if we presume the Poisson braket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \tag{20}$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63 \times 10^{-34}\ \text{J·s} \tag{21}$$

is the Planck's constant with dimensions of action.