(Take Home) Final Exam (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Friday, 2019 May 10, 12:15pm

1. (**50 points.**) Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{1}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{2}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{3}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{4}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\text{min}}}^{r_{\text{max}}} d\phi\right] - 2\pi, \tag{5}$$

where

$$r_{\text{min}} = \frac{r_0}{1+e} \tag{6}$$

is the perihelion, when the planet is closest to Sun, and

$$r_{\text{max}} = \frac{r_0}{1-e} \tag{7}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun.

(a) For the Kepler problem derive the relation

$$d\phi = \frac{L_z dr}{\mu r^2} \frac{1}{\sqrt{\frac{2E}{\mu} + \frac{2\alpha}{\mu r} - \frac{L_z^2}{\mu^2 r^2}}}. \tag{8}$$

Show that the precession of perihelion is zero for the Kepler problem.

(b) When a small correction

$$\delta U(r) = \frac{\beta}{r^2} \tag{9}$$

is added to the potential energy U the precession of the perihelion is no longer zero. Expanding in powers of δU the precession of the perihelion to the leading order for this correction can be evaluated. Verify the relation

$$-\mu\frac{\partial}{\partial L_z}\sqrt{\frac{2E}{\mu}+\frac{2\alpha}{\mu r}-\frac{L_z^2}{\mu^2 r^2}-\frac{2\beta}{\mu r^2}} = \frac{L_z}{\mu r^2}\frac{1}{\sqrt{\frac{2E}{\mu}+\frac{2\alpha}{\mu r}-\frac{L_z^2}{\mu^2 r^2}-\frac{2\beta}{\mu r^2}}}. \tag{10}$$

Show that for the perturbed Kepler problem, to the leading order in the perturbation, we have

$$\Delta\phi = 2\frac{\partial}{\partial L_z}\frac{\mu\alpha}{L_z}\int_{r_{\min}}^{r_{\max}} dr \frac{\frac{\mu\beta}{L_z^2}\frac{r_0^2}{r^2}}{\sqrt{e^2-\left(1-\frac{r_0}{r}\right)^2}}. \tag{11}$$

Thus, evaluate the precession of perihelion. (Hint: Substitute $r_0/r = 1 + e\cos\phi'$.) Show that

$$\Delta\phi = -2\pi\frac{\mu\beta}{L_z^2}. \tag{12}$$

2. (**50 points.**) Refer to the essay by J. M. Luttinger titled 'On "negative" mass in the theory of gravitation' in 1951. (Available on the Internet using a simple search.)

   (a) Reprooduce all the equations in the essay.

   (b) Critically assess the logic of the arguments in the essay.