Homework No. 03 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Tuesday, 2019 Feb 12, 4.30pm

1. (**20 points.**) (Refer Goldstein, 2nd edition, Chapter 1 Problem 8.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{1}$$

in terms of a Lagrangian $L(x, \dot{x}, t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation then

$$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{dF(x,t)}{dt}, \tag{2}$$

where $F(x, t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.

2. (**20 points.**) A mass m_1 is forced to move on a vertical circle of radius R with uniform angular speed ω. Another mass m_2 is connected to mass m_1 using a massless rod of length a, such that it is a simple pendulum with respect to mass m_1. Motion of both the masses is constrained to be in a vertical plane in a uniform gravitational field.

   (a) Write the Lagrangian for the system.

   (b) Determine the equation of motion for the system.

   (c) Give physical interpretation of each term in the equation of motion.

3. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 3. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

   (a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2a^2\dot{\theta}^2 + m_2a\dot{x}\dot{\theta}\cos\theta + m_2ga\cos\theta. \tag{3}$$

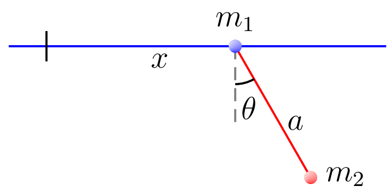

Figure 1: Problem 3.

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos\theta, \tag{4a}$$

$$\frac{\partial L}{\partial x} = 0, \tag{4b}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos\theta, \tag{4c}$$

$$\frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin\theta - m_2 g a \sin\theta. \tag{4d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2 \sin\theta = 0, \tag{5a}$$

$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0. \tag{5b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a\ddot{\theta} = 0, \tag{6a}$$

$$a\ddot{\theta} + \ddot{x} + g\theta = 0. \tag{6b}$$

Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \tag{7}$$

Interpret this solution.

(e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (5), using Eq. (5a) in Eq. (5b), as

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2 \sin\theta = 0, \tag{8a}$$

$$\sin\theta \Big[a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta + g \Big] = 0. \tag{8b}$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpretting Eqs. (8) let us identify the coordinates of the center of mass of the m_1-m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1 x + m_2(x + a\sin\theta), \tag{9a}$$
$$(m_1 + m_2)y_{\text{cm}} = -m_2 a\cos\theta, \tag{9b}$$

which for $m_1 = 0$ are the coordinates of the mass m_2,

$$x_{\text{cm}} = x + a\sin\theta, \tag{10a}$$
$$y_{\text{cm}} = -a\cos\theta. \tag{10b}$$

Show that

$$\dot{x}_{\text{cm}} = \dot{x} + a\dot{\theta}\cos\theta, \tag{11a}$$
$$\dot{y}_{\text{cm}} = a\dot{\theta}\sin\theta, \tag{11b}$$

and

$$\ddot{x}_{\text{cm}} = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{12a}$$
$$\ddot{y}_{\text{cm}} = a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta. \tag{12b}$$

Comparing Eqs. (8) and Eqs. (12) we learn that

$$\ddot{x}_{\text{cm}} = 0, \tag{13a}$$
$$\sin\theta\Big[\ddot{y}_{\text{cm}} + g\Big] = 0. \tag{13b}$$

Thus, $\ddot{y}_{\text{cm}} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

(f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$. Show that for this case $\dot{x}_{\text{cm}}(0) = 0$ and

$$a(\cos\theta - \cos\theta_0) = \frac{1}{2}gt^2. \tag{14}$$

Plot θ as a function of time t. Interpret this solution.