Homework No. 04 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Thursday, 2019 Feb 28, 4.30pm

1. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

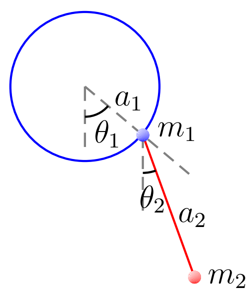

Figure 1: Problem 1.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1,\dot\theta_1,\theta_2,\dot\theta_2) = \frac{1}{2}m_2a_1^2\dot\theta_1^2+\frac{1}{2}m_2a_2^2\dot\theta_2^2+m_2a_1a_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)+m_2ga_1\cos\theta_1+m_2ga_2\cos\theta_2. \tag{1}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial\dot\theta_1} = m_2a_1^2\dot\theta_1 + m_2a_1a_2\dot\theta_2\cos(\theta_1-\theta_2), \tag{2a}$$

$$\frac{\partial L}{\partial\theta_1} = -m_2a_1a_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - m_2ga_1\sin\theta_1, \tag{2b}$$

$$\frac{\partial L}{\partial\dot\theta_2} = m_2a_2^2\dot\theta_2 + m_2a_1a_2\dot\theta_1\cos(\theta_1-\theta_2), \tag{2c}$$

$$\frac{\partial L}{\partial\theta_2} = m_2a_1a_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - m_2ga_2\sin\theta_2. \tag{2d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta}\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{3a}$$

$$\ddot{\theta}_2 + \omega_2^2 \sin\theta_2 + \beta\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{3b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{4}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 = 0, \tag{5a}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0. \tag{5b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0. \tag{6}$$

Interpret and expound your solution.

2. (**20 points.**) Consider the coplanar double pendulum in Figure 2.

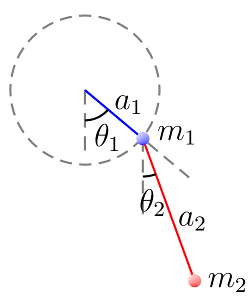

Figure 2: Problem 2.

(a) Write the Lagrangian for the system. in particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \tag{7}$$

where

$$L_1 = \frac{1}{2}(m_1 + m_2)a_1^2\dot{\theta}_1^2 + (m_1 + m_2)ga_1\cos\theta_1, \tag{8a}$$

$$L_2 = \frac{1}{2}m_2 a_2^2\dot{\theta}_2^2 + m_2 g a_2 \cos\theta_2, \tag{8b}$$

$$L_{\text{int}} = m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2). \tag{8c}$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1+m_2)a_1\ddot{\theta}_1 + (m_1+m_2)g\sin\theta_1 + m_2a_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2a_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0, \quad \text{(9a)}$$

$$a_2\ddot{\theta}_2 + g\sin\theta_2 + a_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - a_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0. \quad \text{(9b)}$$

(c) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{\alpha}{\beta}\ddot{\theta}_2 = 0, \quad \text{(10a)}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_1 + \beta\ddot{\theta}_1 = 0, \quad \text{(10b)}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \alpha = \frac{m_2}{m_1+m_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \quad \text{(11)}$$

Note that $0 \leq \alpha \leq 1$.

(d) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = \omega_0, \quad \text{(12)}$$

for $\alpha = 1/2$ and $\beta = 1$.