Homework No. 06 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Tuesday, 2019 Mar 26, 4.30pm

1. (**40 points.**) In terms of the Lagrangian function $L(\mathbf{r}, \mathbf{v}, t)$ the action functional $W[\mathbf{r}; t_1, t_2]$ is defined as

$$W[\mathbf{r}; t_1, t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r}, \mathbf{v}, t), \tag{1}$$

where $\mathbf{v} = d\mathbf{r}/dt$.

(a) For arbitrary infinitesimal variations in the path

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t), \tag{2}$$

and infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \tag{3}$$

the change in action is given by

$$\begin{aligned} \delta W = &\int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ &+ \int_{t_1}^{t_2} dt \left[\delta t\left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) + \delta\mathbf{r}\cdot\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right)\right], \end{aligned} \tag{4}$$

where the canonical momentum and the Hamiltonian are defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} \quad \text{and} \quad H = \mathbf{v}\cdot\mathbf{p} - L \tag{5}$$

respectively.

(b) The change in the action due to variations in path is captured in the functional derivative

$$\frac{\delta W}{\delta \mathbf{r}(t)} = \left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right) + \Big[\delta(t - t_2) - \delta(t - t_1)\Big]\mathbf{p}. \tag{6}$$

The change in the action due to time transformation is captured in the functional derivative

$$\frac{\delta W}{\delta t(t)} = \left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) - \Big[\delta(t - t_2) - \delta(t - t_1)\Big]H. \tag{7}$$

(c) In terms of the Hamiltonian the action takes the form

$$W[\mathbf{r}, \mathbf{p}; t_1, t_2] = \int_{t_1}^{t_2} dt\Big[\mathbf{v}\cdot\mathbf{p} - H(\mathbf{r}, \mathbf{p}, t)\Big]. \tag{8}$$

(d) Show that for for arbitrary infinitesimal variations in coordinate and momentum

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t) \quad \text{and} \quad \bar{\mathbf{p}}(t) = \mathbf{p}(t) - \delta\mathbf{p}(t), \tag{9}$$

and infinitesimal general time transformation, the change in action is given by

$$\delta W = \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] + \int_{t_1}^{t_2} dt \left[\delta t\left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right) - \delta\mathbf{r}\cdot\left(\frac{d\mathbf{p}}{dt} + \frac{\partial H}{\partial \mathbf{r}}\right) + \delta\mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}}\right)\right]. \tag{10}$$

2. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\omega}\times\mathbf{r}, \quad \delta\mathbf{p} = \delta\boldsymbol{\omega}\times\mathbf{p}, \quad \delta t = 0, \tag{11}$$

where $d\delta\boldsymbol{\omega}/dt = 0$. Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r}\times\frac{\partial H}{\partial\mathbf{r}} + \mathbf{p}\times\frac{\partial H}{\partial\mathbf{p}}\right] = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{12}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is the angular momentum. Thus, state the conditions to be satisfied by the Hamitonian for the angular momentum $\mathbf{L}$ of the system to be conserved.

3. (**30 points.**) (Refer Schwinger, chapter 9) The Hamiltonian for a hydrogenic atom is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Ze^2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{13}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2 and charges e and Ze.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1+m_2}, \tag{14}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{Ze^2}{r}, \tag{15}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{16}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -Ze^2\frac{\mathbf{r}}{r^3}. \tag{17}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Ze^2}\mathbf{p}\times\mathbf{L}, \tag{18}$$

are the three constants of motion for a hydrogenic atom.