

Homework No. 07 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Tuesday, 2019 Apr 9, 4.30pm

1. **(20 points.)** (Refer Landau and Lifshitz, Problem 1 in Chapter 3.)

A simple pendulum, consisting of a particle of mass m suspended by a string of length l in a uniform gravitational field g , is described by the Hamiltonian

$$H = \frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos \phi. \quad (1)$$

- (a) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

$$\frac{1}{2}ml^2\dot{\phi}^2 - mgl \cos \phi = -mgl \cos \phi_0. \quad (2)$$

Thus, derive

$$\frac{dt}{T_0} = \frac{1}{2\pi} \frac{d\phi}{\sqrt{2(\cos \phi - \cos \phi_0)}} \quad (3)$$

where $T_0 = 2\pi\sqrt{l/g}$.

- (b) Determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K \left(\sin \frac{\phi_0}{2} \right), \quad (4)$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (5)$$

is the complete elliptic integral of the first kind.

- (c) Using the power series expansion

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 k^{2n} \quad (6)$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0 \left[1 + \frac{\phi_0^2}{16} + \dots \right]. \quad (7)$$

- (d) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60^\circ$.

(e) Plot the time period T of Eq. (4) as a function of ϕ_0 . What can you conclude about the time period for $\phi_0 = \pi$?

2. (20 points.) Starting from the Lagrangian for the Kepler problem,

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}, \quad (8)$$

derive Kepler's first law of planetary motion, which states that the orbit of a planet is a conic section. In particular, derive

$$r(\phi) = \frac{r_0}{1 + e \cos(\phi - \phi_0)}, \quad (9)$$

which is the equation of a conic section in terms of the eccentricity e and a distance r_0 . The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \quad (10)$$

is minimum at r_0 . We used the definitions, $L_z = \mu r^2 \dot{\phi}$,

$$r_0 = \frac{L_z^2}{\mu\alpha}, \quad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \quad e = \sqrt{1 + \frac{E}{U_{\text{eff}}(r_0)}}. \quad (11)$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z , which are constants of motion.