Homework No. 08 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Tuesday, 2019 Apr 16, 4.30pm

1. (**50 points.**) The Hamiltonian for a Kepler problem is

$$H(\mathbf{r},\mathbf{p}) = \frac{p^2}{2\mu} - \frac{\alpha}{r}. \tag{1}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and the axial vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times\mathbf{L}}{\mu\alpha}, \tag{2}$$

are conserved quantities for a Kepler problem.

(a) Show that

$$\mathbf{W} = \frac{\mu\alpha}{L^2}\mathbf{A}\times\mathbf{L} \tag{3}$$

is also a conserved quantity. That is, show that $d\mathbf{W}/dt = 0$. Thus, together, the vectors $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{W}$, form an orthogonal set that remain fixed in time. Show that the vector $\mathbf{W}$ can be expressed in the form

$$\mathbf{W} = \mathbf{p} + \frac{\mu\alpha}{L^2}\hat{\mathbf{r}}\times\mathbf{L}. \tag{4}$$

Further, show that

$$W = \mu\alpha\frac{A}{L}. \tag{5}$$

(b) Determine the components of the momentum $\mathbf{p}$ along these orthogonal vectors by evaluating $(\mathbf{p}\cdot\hat{\mathbf{L}})$, $(\mathbf{p}\cdot\hat{\mathbf{A}})$, and $(\mathbf{p}\cdot\hat{\mathbf{W}})$. Thus, construct the momentum $\mathbf{p}$ in the form

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{L}})\,\hat{\mathbf{L}} + (\mathbf{p}\cdot\hat{\mathbf{A}})\,\hat{\mathbf{A}} + (\mathbf{p}\cdot\hat{\mathbf{W}})\,\hat{\mathbf{W}}. \tag{6}$$

Hint: Show that

$$\mathbf{p}\cdot\mathbf{L} = 0, \qquad \mathbf{p}\cdot\mathbf{A} = \mathbf{p}\cdot\hat{\mathbf{r}}, \qquad \mathbf{p}\cdot\mathbf{W} = \frac{p^2}{2} + \mu H. \tag{7}$$

(c) It is well known that the position $\mathbf{r}$ traverses an ellipse about the origin. This is the content of Kepler's first law of motion. Show that the momentum $\mathbf{p}$ traverses a circle about a fixed point $\mathbf{p}_0$. That is, show that the momentum $\mathbf{p}$ satisfies the equation of a circle,

$$|\mathbf{p} - \mathbf{p}_0| = q. \tag{8}$$

Hint: Rewrite the expression for $(\mathbf{p}\cdot\hat{\mathbf{W}})$ in the form $\mathbf{p}\cdot\mathbf{p} - 2\mathbf{p}\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{W} = W^2 - 2\mu H$.

(d) Determine the vector $\mathbf{p}_0$ representing the center of this circle, and find the radius q of this circle. Verify that the center $\mathbf{p}_0$ is a conserved quantity.
Solution: $\mathbf{p}_0 = \mathbf{W}$ and $q = \mu\alpha/L$.

(e) Show that when the position $\mathbf{r}$ traverses a circle ($A = 0$) the center of the circle traversed by momentum $\mathbf{p}$ is the origin.