Homework No. 10 (Spring 2019)

PHYS 510: Classical Mechanics

Due date: Thursday, 2019 May 2, 4.30pm

1. (**100 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{1}$$

Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector

$$x^\alpha = (ct, \mathbf{x}). \tag{2}$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{3}$$

i. Using Eq. (1) deduce

$$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{4}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{5}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{6}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{7}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{8}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{9a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{10}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{11}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{12}$$

i. Show that

$$a^\alpha = \gamma\left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a}\right), \tag{13}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{14}$$

is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (6) and taking derivative with respect to proper time show that

$$u^\alpha a_\alpha = 0. \tag{15}$$

Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (15) derive the identity

$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\gamma^3. \tag{16}$$

iv. Show that

$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4\right) \tag{17}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where

$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{18}$$

is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration

$$a^\alpha a_\alpha = \mathbf{a}_0 \cdot \mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)^2 \gamma^2\right]\gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c}\right)^2\right]\gamma^6. \tag{19}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{20}$$

And, in a particular frame, if $\mathbf{v}\perp\mathbf{a}$ (corresponding to circular motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{21}$$

(d) Force: The force four-vector is defined as

$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma\right), \tag{22}$$

where the force $\mathbf{F}$, identified (or defined) as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{23}$$

is the force in Newtonian physics. Starting from Eq. (10) derive the relation

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{24}$$

which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = ma^\alpha. \tag{25}$$

Show that these involve the relations

$$\mathbf{F} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{26a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{26b}$$

Discuss the non-relativistic limits of the equations of motion.

2. (**30 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{27}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{28}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 2, and the world-line of particle 2 is the red curve in Figure 2. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

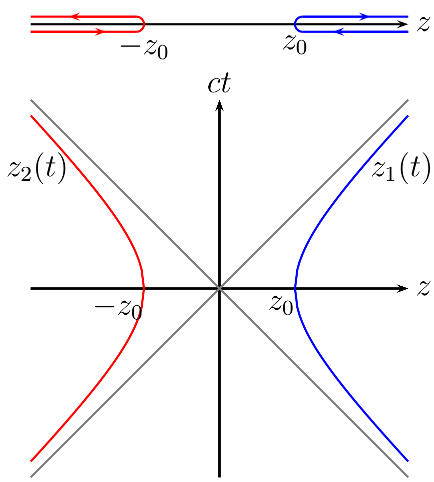

Figure 1: Problem 2

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?