(Take Home) Final Exam (Spring 2019)

PHYS 520B: Electromagnetic Theory

Due date: Thursday, 2019 May 9, 12.15pm

Electromagnetic scattering can be broadly classified as elastic and inelastic. In an elastic scattering the obstacle does not absorb or dissipate energy. Rayleigh scattering, Thompson scattering, and Mie scattering are examples of elastic scattering. Thompson scattering is the regime when the energy of the incident wave is large relative to the characteristic energy (say the rest mass energy) of the obstacle, and thus is further classified as high energy scattering. In this spirit Rayleigh scattering is a low energy scattering, the energy of the incident wave is small in comparison to the characteristic energy of the obstacle. Energies in Mie scattering are intermediate between Rayleigh and Thompson scattering. In contrast, in an inelastic scattering the obstacles absorb or dissipate energy. Raman scattering is the inelastic version of low energy Rayleigh scattering, and Compton scattering is the inelastic version of the high energy Thompson scattering.

	Elastic	Inelastic
Low energy	Rayleigh	Raman
High energy	Thompson	Compton

Table 1: A simple classification of scattering processes

Obstacles and materials are in general neutral. Thus, the interaction of electromagnetic fields and charges is not the dominant effect in scattering. Instead, for neutral materials, the electromagnetic interactions are characterized by electric and magnetic dipole moments. We shall not include magnetic properties in this discussion, for simplicity, and because magnetic effects are most often subdominant. The scattering mechanism involves an incoming electromagnetic wave with oscillating electric and magnetic fields that induce oscillations in the electric properties of the obstacle which then radiates electromagnetic waves and the sum of all the individual radiations is observed as the scattered wave. The macroscopic quantity that characterizes the induced polarization is called the electric polarizability of the materials or the dielectric constant $\boldsymbol{\varepsilon}(\mathbf{r},\omega)$ of the material, which is a frequency depedent property. The electromagnetic interactions of electrically polarizable materials are governed by the macroscopic Maxwell equations. In the Fourier transformed frequency domain the Maxwell equations, in the absence of charges and currents, in SI units, are

$$\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r},\omega) = i\omega \mathbf{B}(\mathbf{r},\omega), \tag{1a}$$

$$\boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r},\omega) = -i\omega \mathbf{D}(\mathbf{r},\omega), \tag{1b}$$

where the constitutive fields are, assuming linear response,

$$\mathbf{D}(\mathbf{r},\omega) = \boldsymbol{\varepsilon}(\mathbf{r},\omega)\cdot\varepsilon_0\mathbf{E}(\mathbf{r},\omega), \tag{2a}$$
$$\mathbf{B}(\mathbf{r},\omega) = \boldsymbol{\mu}(\mathbf{r},\omega)\cdot\mu_0\mathbf{H}(\mathbf{r},\omega). \tag{2b}$$

1. Verify that, in the absence of charges and currents, valid for neutral materials, the constitutive fields $\mathbf{D}$ and $\mathbf{B}$ are divergenceless, which are implicit in Eqs. (1) and verified by taking divergence in these equations. Show that, the Maxwell equations, together, imply the dyadic differential equation for the electric field

$$\left[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1}) - \frac{\omega^2}{c^2}\mathbf{1}\right]\cdot\mathbf{E}(\mathbf{r},\omega) = \frac{\omega^2}{c^2}\big[\boldsymbol{\varepsilon}(\mathbf{r},\omega)-\mathbf{1}\big]\cdot\mathbf{E}(\mathbf{r},\omega). \tag{3}$$

The magnetic field is then given in terms of the electric field using Eq. (1a). The dyadic differential equation in Eq. (3) in principle governs all phenomena involving electromagnetic fields and electrically polarizable materials. A particular phenomenon of interest is chosen, out of the multitude of processes governed by this dyadic equation, by specifying the boundary conditions satisfied by the fields. In this manner, an electromagnetic scattering process is characterized by the boundary conditions

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0(\mathbf{r},\omega) + \mathbf{K}(\theta,\phi,\omega)\,\frac{e^{i\frac{\omega}{c}r}}{r}. \tag{4}$$

These boundary conditions are dictated by the requirement that the electric field $\mathbf{E}(\mathbf{r},\omega)$ has a component that involves an incident wave $\mathbf{E}_0(\mathbf{r},\omega)$, which is the part of field that is independent of the obstacle, and another component that involves a spherical outgoing wave. Note that in scattering problems we do not require $\mathbf{E}(\mathbf{r},\omega)$ to go to zero in the far-field regions at $r\to\infty$. In fact, the scattering amplitude

$$\mathbf{K}(\hat{\mathbf{r}},\omega) \equiv \mathbf{K}(\theta,\phi,\omega) \tag{5}$$

in Eq. (4), which is the central quantity of interest in a scattering problem is completely contained in the asymptotic $r\to\infty$ part of $\mathbf{E}(\mathbf{r},\omega)$.

2. Let the incident wave be a monochromatic plane wave of frequency ω with fields in time domain given by

$$\mathbf{E}_0(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{6a}$$
$$\mathbf{B}_0(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{6b}$$

Since $\mathbf{E}_0(\mathbf{r},t)$ satisfy the Maxwell equations, independently, we have

$$k = \frac{\omega}{c}, \tag{7}$$

and the direction of $\mathbf{k}$ is constrained by

$$\varepsilon_0\mathbf{E}_0^*\times\mathbf{B}_0 = \hat{\mathbf{k}}\frac{U}{c}, \tag{8}$$

with $\mathbf{k}\cdot\mathbf{E}_0 = 0$ and $\mathbf{k}\cdot\mathbf{B}_0 = 0$, U being the electromagnetic energy density. Show that, in the frequency domain the incident field is given by

$$\mathbf{E}_0(\mathbf{r},\omega) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9a}$$
$$\mathbf{B}_0(\mathbf{r},\omega) = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{9b}$$

3. The formal solution to the dyadic differential equation in Eq. (3) can be expressed as an integral equation

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_0(\mathbf{r},\omega) - (\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1})\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\cdot\big[\boldsymbol{\varepsilon}(\mathbf{r}',\omega) - \mathbf{1}\big]\cdot\mathbf{E}(\mathbf{r}',\omega), \tag{10}$$

which need not be proved here. The scattering boundary conditions given by Eq. (4) require us to impose the far-field approximation $r' \ll r$ which amounts to the replacement

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} = r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) + \mathcal{O}\left(\frac{r'}{r}\right)^2. \tag{11}$$

Thus, in the far-field asymptotic limit we can replace

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{1}{4\pi}\frac{e^{ikr}}{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{12}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{13}$$

In this form we see the structure of the spherical outgoing wave e^{ikr}/r emerging. Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla}\frac{e^{ikr}}{r} \to i\mathbf{k}'\,\frac{e^{ikr}}{r} \tag{14}$$

Show that these lead to the dyadic transcription

$$-(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1})\frac{e^{ikr}}{r} = (\mathbf{k}'\mathbf{k}' - \frac{\omega^2}{c^2}\mathbf{1})\frac{e^{ikr}}{r} = \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})k^2\frac{e^{ikr}}{r}, \tag{15}$$

where $\mathbf{1}$ is the unit dyadic. Thus, derive the following expression for the scattering amplitude

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})\frac{k^2}{4\pi}\int d^3r'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}\cdot\big[\boldsymbol{\varepsilon}(\mathbf{r}',\omega) - \mathbf{1}\big]\cdot\mathbf{E}(\mathbf{r}',\omega). \tag{16}$$

Starting from Eq. (1a) derive the relation

$$c\mathbf{B}(\mathbf{r},\omega) = c\mathbf{B}_0(\mathbf{r},\omega) + \hat{\mathbf{r}}\times\mathbf{K}(\hat{\mathbf{r}},\omega)\frac{e^{i\frac{\omega}{c}r}}{r}, \tag{17}$$

where

$$\hat{\mathbf{r}} \times \mathbf{K}(\hat{\mathbf{r}}, \omega) = -\hat{\mathbf{r}} \times \omega^2 \frac{\mu_0}{4\pi} \int d^3 r' \, e^{-i\mathbf{k}' \cdot \mathbf{r}'} \cdot \big[\boldsymbol{\varepsilon}(\mathbf{r}', \omega) - \mathbf{1} \big] \cdot \varepsilon_0 \mathbf{E}(\mathbf{r}', \omega). \tag{18}$$

Observe that the far-field approximation is a means to impose the boundary condition for a scattering process, and thus is a defining property of the scattering problem. We recognize that the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ is evaluated in the far-field asymptotic region ($r \to \infty$) while the contributions to the scattering amplitude, as per Eq. (16), comes from short-range where $[\boldsymbol{\varepsilon}(\mathbf{r}, \omega) - \mathbf{1}]$ is nonzero, that is, from regions inside the obstacle. To summarize, in this section, we have formulated the scattering problem in terms of the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ given by Eq. (16) in which the electric field is determined by solving the dyadic integral equation in Eq. (10) satisfying boundary conditions in Eq. (4). We add that in general this is a formidable problem and further progress requires us to use approximations, that in turn limits the validity of the solutions.

4. The total energy E radiated into the solid angle $d\Omega$ per unit (positive, $0 \leq \omega < \infty$) frequency range $d\omega$ is given by

$$\frac{\partial}{\partial \omega} \frac{\partial E}{\partial \Omega} = \frac{1}{\pi} \frac{r^2}{c\mu_0} \Big| c\mathbf{B}(\mathbf{r}, \omega) \Big|^2. \tag{19}$$

Show that

$$\frac{\partial}{\partial \omega} \frac{\partial E}{\partial \Omega} = \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi} \right) \frac{1}{\pi} \left| \frac{\hat{\mathbf{r}} \times \mathbf{K}(\hat{\mathbf{r}}, \omega)}{c \frac{\mu_0}{4\pi}} \right|^2. \tag{20}$$

Verify that the quantity inside absolute value sign has the dimensions of charge. (Caution: Fourier transform changes dimensions of quantities.)

Rayleigh scattering off a point obstacle

Using the boundary condition in Eq. (4) iteratively in the expression for the scattering amplitude in Eq. (16) we obtain a series that is convergent when

$$\frac{k^2 \alpha}{r} < 1, \tag{21}$$

where $\boldsymbol{\alpha} = \int d^3 r \, [\boldsymbol{\varepsilon}(\mathbf{r}) - \mathbf{1}] \sim l^3$ is the measure of the size of the obstacle. Since we necessarily have $l \ll r$ for the far-field regions we have the requirement for the convergence to be

$$l \ll \lambda. \tag{22}$$

That is the dimensions of the obstacle are significatly smaller than the wavelength of the incident plane wave. This is the Rayeigh scattering approximation. In practice, the expression for the scattering amplitude in the extreme limit of Rayleigh scattering is obtained with the following replacements in Eq. (16):

$$\mathbf{E}(\mathbf{r}) \to \mathbf{E}_0(\mathbf{r}), \tag{23a}$$

$$[\boldsymbol{\varepsilon}(\mathbf{r}) - \mathbf{1}] \to 4\pi \boldsymbol{\alpha}(\mathbf{s}) \, \delta^{(3)}(\mathbf{r} - \mathbf{s}), \tag{23b}$$

where $\mathbf{s}$ is the position of the obstacle, that has been, for convenience, modeled as a point obstacle using δ-functions. Here $\boldsymbol{\alpha}$ is the polarizability of the obstacle, and has dimensions of length-cube. Show that these replacements lead to the Rayleigh scattering amplitude

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \boldsymbol{\alpha}(\mathbf{s}) \cdot \mathbf{E}_0(\mathbf{s})]\, k^2\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{s}}. \tag{24}$$

For obstacles with isotropic polarizabilities we have $\boldsymbol{\alpha}(\mathbf{s}) = \mathbf{1}\alpha(\mathbf{s})$ and the scattering amplitude takes the form

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \mathbf{E}_0(\mathbf{s})]\, k^2\, \alpha(\mathbf{s}) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{s}}. \tag{25}$$

Rayleigh scattering off a thin film

If the obstacles are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$,

$$[\boldsymbol{\varepsilon}(\mathbf{r}) - \mathbf{1}] \to 4\pi\boldsymbol{\lambda}(\mathbf{s})\, \delta(z), \tag{26}$$

where the δ-function has been used to describe the assumption that the obstacles in a thin film are confined to a plane, $z = 0$ here. Once the obstacles are restricted to be on a plane, we can choose the direction of incidence $\mathbf{k}$ of the plane wave to be normal to the plane. That is, $\mathbf{k} \cdot \mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point obstacles on the plane. Further, notice that in this special case the electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations the Rayleigh scattering amplitude is given by, for isotropic polarizabilities,

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \mathbf{K}_0\, k^2 \int d^2s\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{s}} \lambda(\mathbf{s}), \tag{27}$$

where

$$\mathbf{K}_0 = \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{E}_0). \tag{28}$$

For a disc of radius R centered at the origin with uniform polarizability per unit area λ we can complete the integrals to obtain

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \mathbf{K}_0\, \lambda k^2 \pi R^2 2 \frac{J_1(kR\sin\theta)}{kR\sin\theta}. \tag{29}$$

Here we used the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{it\cos\phi} \tag{30}$$

and the identity

$$\int_0^b t dt J_0(t) = bJ_1(b). \tag{31}$$

Note the limiting value

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2}, \tag{32}$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.