Midterm Exam No. 01 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Date: 2019 Feb 19

1. (**20 points.**) Consider a straight wire of radius a carrying current I described using the current density
$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}}\,\frac{C}{\rho}\,e^{-\lambda\rho}\,\theta(a-\rho), \tag{1}$$
where $\theta(x) = 1$ for $x > 1$ and zero otherwise.

   (a) Find C in terms of the current I.

   (b) Find the magnetic field inside and outside the wire.

   (c) Plot the magnetic field as a function of ρ.

2. (**20 points.**) Very briefly (in a couple of sentences) describe Meissner effect in a superconductor.

3. (**20 points.**) A circular loop of wire carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

   (a) Show that the current density generated by this motion is given by
$$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(\rho-a)\delta(z-0). \tag{2}$$
Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

   (b) Using
$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{3}$$
determine the magnetic dipole moment of this loop to be
$$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{4}$$

   (c) Calculate the angular momentum of the rotating loop to be
$$\mathbf{L} = ma^2\boldsymbol{\omega}, \tag{5}$$
where m is the mass of the loop.

   (d) What is the gyromagnetic ratio g of the rotating loop, which is defined by the relation $\mathbf{m} = g\mathbf{L}$.

4. (**20 points.**) It is perplexing that the magnetic field due to an infinitely long solenoid is independent of the radius of the solenoid. It prompts us to investigate what would happen if we take the radius of the solenoid to be small, limiting all the way to zero. With such a configuration in mind let us consider the following magnetic vector potential

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{C}{\rho}\,\hat{\boldsymbol{\phi}}, \tag{6}$$

where C is constant. What can we conclude about the dimensions of C, say, in terms of magnetic moment. Note that we could imagine a solenoid of infinitely thin radius to be built of point magnetic moments stacked on the z axis.

(a) Evaluate the magnetic field for this magnetic vector potential using the relation

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{7}$$

In particular, show that

$$\mathbf{B} = 0 \quad \text{for} \quad \rho \neq 0. \tag{8}$$

(b) Using Stoke's theorem deduce

$$\int_S d\mathbf{S} \cdot \mathbf{B} = \oint_S d\mathbf{l} \cdot \mathbf{A}, \tag{9}$$

where S is a surface. Let S be a planar surface perpendicular to the z axis and crossing it. Is there an inconsistency? Show that the inconsistency is avoided by having the magnetic field to be

$$\mathbf{B} = \hat{\mathbf{z}}\,\frac{\mu_0}{2} C \delta^{(2)}(\boldsymbol{\rho}). \tag{10}$$

(c) Evaluate

$$\boldsymbol{\nabla} \cdot \mathbf{B}. \tag{11}$$

Is it zero?