Homework No. 03 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Due date: Tuesday, 2019 Feb 19, 4.30pm

1. (**40 points.**) The current density for a circular loop of radius a carrying a steady current I is given by
$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I\delta(\rho - a)\delta(z), \tag{1}$$
where the the loop is chosen to be in the x-y plane with the origin as its center.

   (a) Using Bio-Savart law and completing the integrals involving δ-functions show that magnetic field has the form
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \frac{\left[a^2\hat{\mathbf{z}} + az\hat{\boldsymbol{\rho}}' - a\rho(\hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{\phi}}')\right]}{[z^2+\rho^2+a^2-2\rho a\cos(\phi-\phi')]^{\frac{3}{2}}}. \tag{2}$$

   (b) Substitute $\phi' - \phi \to \phi'$ and show that
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi' \frac{\left[(a^2 - a\rho\cos\phi')\hat{\mathbf{z}} + az\hat{\boldsymbol{\rho}}\cos\phi' + az\hat{\boldsymbol{\phi}}\sin\phi'\right]}{[z^2+\rho^2+a^2-2\rho a\cos\phi']^{\frac{3}{2}}}. \tag{3}$$

   (c) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,
$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1-k^2\sin^2\psi}}, \tag{4a}$$
$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1-k^2\sin^2\psi}, \tag{4b}$$
respectivly, the magnetic field is
$$\begin{aligned}\mathbf{B}(\mathbf{r}) = {}& \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\left[K(k) - \frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2}E(k)\right] \\ & -\hat{\boldsymbol{\rho}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\frac{z}{\rho}\left[K(k) - \frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2}E(k)\right],\end{aligned} \tag{5}$$
where
$$k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}. \tag{6}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (3) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a\cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals. It is useful to identify

$$\int_0^{\frac{\pi}{2}} d\psi \frac{1}{(1-k^2\sin^2\psi)^{\frac{3}{2}}} = \frac{E(k)}{(1-k^2)}. \tag{7}$$

2. A charged spherical shell carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter, say z-axis.

   (a) Show that the current density generated by this motion is given by

   $$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(r-a). \tag{8}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

   (b) Using

   $$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{9}$$

   determine the magnetic dipole moment of the rotating sphere to be

   $$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{10}$$

   (c) Calculate the vector potential $\mathbf{A}(\mathbf{r})$ inside and outside the sphere, without choosing $\mathbf{r}$ to be along $\hat{\mathbf{z}}$.
   Hint: Observe that

   $$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}} \\ &= \frac{1}{2}\sqrt{\frac{8\pi}{3}}\Big[-Y_{11}(\theta,\phi)+Y_{1,-1}(\theta,\phi)\Big]\hat{\mathbf{i}} + \frac{1}{2i}\sqrt{\frac{8\pi}{3}}\Big[-Y_{11}(\theta,\phi)-Y_{1,-1}(\theta,\phi)\Big]\hat{\mathbf{j}} \\ &\quad+\sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi)\hat{\mathbf{k}}.\end{aligned} \tag{11}$$