

Homework No. 04 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Due date: Tuesday, 2019 Mar 12, 4.30pm

1. **(20 points.)** Using Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{E}(\mathbf{r}, t) = -\frac{1}{\varepsilon_0} \nabla \rho(\mathbf{r}, t) - \frac{1}{\varepsilon_0} \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}, t), \quad (1a)$$

$$\left(-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \mathbf{B}(\mathbf{r}, t) = \mu_0 \nabla \times \mathbf{J}(\mathbf{r}, t). \quad (1b)$$

2. **(20 points.)** The following form for the electric and magnetic field

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}, \quad (2a)$$

$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t + i\delta}, \quad (2b)$$

involves a phase difference δ between the electric and magnetic field strength. Here \mathbf{E}_0 and \mathbf{B}_0 are constants. Assume no charges or currents. Investigate if such a solution is permitted by the Maxwell equations?

3. **(80 points.)** The 4-dimensional Euclidean Green's function satisfies

$$-\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_4^2}\right) G_E(x_1, \dots, x_4) = \delta(x_1) \cdots \delta(x_4). \quad (3)$$

- (a) Show that the solution to this equation can be written as the Fourier transform

$$G_E(x_1, \dots, x_4) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \frac{e^{i(k_1 x_1 + \dots + k_4 x_4)}}{k_1^2 + \dots + k_4^2}. \quad (4)$$

- (b) Verify the integral

$$\frac{1}{M} = \int_0^{\infty} ds e^{-sM}. \quad (5)$$

- (c) Using Eq. (5) in Eq. (4) show that

$$G_E(x_1, \dots, x_4) = \int_0^{\infty} ds \prod_{m=1}^4 \left[\int_{-\infty}^{\infty} \frac{dk_m}{2\pi} e^{-sk_m^2 + ik_m x_m} \right]. \quad (6)$$

(d) Show that

$$\int_{-\infty}^{\infty} dk_m e^{-sk_m^2 + ik_m x_m} = \sqrt{\frac{\pi}{s}} e^{-\frac{x_m^2}{4s}} \quad (7)$$

(e) Using the integral of Eq. (7) in Eq. (6) and using the integral representation of Gamma function,

$$\Gamma(z) = \int_0^{\infty} \frac{ds}{s} s^z e^{-s}, \quad (8)$$

show that

$$G_E(x_1, \dots, x_4) = \frac{1}{4\pi^2} \frac{1}{x_1^2 + \dots + x_4^2}. \quad (9)$$

(f) By making the complex replacement

$$x_4 \rightarrow ict \equiv \lim_{\varepsilon \rightarrow 0+} e^{i(\frac{\pi}{2} - \varepsilon)} ct, \quad (10)$$

show that

$$D_+(\mathbf{r}, t) = iG_E(\mathbf{r}, ict) \quad (11)$$

satisfies the differential equation

$$\left(-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) D_+(\mathbf{r}, t) = \delta^{(3)}(\mathbf{r}) \delta(ct), \quad (12)$$

where $\mathbf{r} = (x_1, x_2, x_3)$, with the corresponding solution

$$D_+(\mathbf{r}, t) = \lim_{\varepsilon \rightarrow 0+} \frac{i}{4\pi^2} \frac{1}{\mathbf{r}^2 - (ct)^2 + i\varepsilon'}, \quad (13)$$

where $\varepsilon' = (ct)^2 \varepsilon$

(g) Using the δ -function representation

$$\pi \delta(x) = \lim_{\varepsilon \rightarrow 0+} \frac{\varepsilon}{x^2 + \varepsilon^2} = \lim_{\varepsilon \rightarrow 0+} \text{Im} \frac{1}{x - i\varepsilon} \quad (14)$$

and the identity

$$\delta(r^2 - (ct)^2) = \frac{1}{2r} [\delta(r - ct) + \delta(r + ct)] \quad (15)$$

show that

$$\text{Re } D_+(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\delta(r - ct)}{4\pi r} + \frac{\delta(r + ct)}{4\pi r} \right], \quad (16)$$

where the two terms here are the retarded and advanced Green's functions, respectively, up to numerical factors.

(h) Refer problem 31.9 in Schwinger et al. for further discussion on this subject. (Will not be graded.)