Homework No. 06 (Spring 2019)

PHYS 520B: Electromagnetic Theory

Due date: Tuesday, 2019 Apr 16, 12.35pm

1. (**60 points.**) The electromagnetic fields,

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{1a}$$
$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{1b}$$

in the Maxwell equations, in SI units,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \tag{2a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{2b}$$
$$-\boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2c}$$
$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{2d}$$

are determined in terms of the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$ by the relations

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{3}$$

These potentials are not uniquely defined, because if we let

$$\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda, \qquad \phi \to \phi - \frac{\partial \lambda}{\partial t}, \tag{4}$$

the electric and magnetic fields in Eq. (3) remain unaltered for an arbitrary function $\lambda = \lambda(\mathbf{r}, t)$. This is called gauge invariance or gauge symmetry. This symmetry allows us to choose a gauge for simplifying a calculation. In the Lorenz gauge,

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \tag{5}$$

the electric scalar potential and the magnetic vector potential are given in terms of inhomogeneous wave equations,

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r}, t) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}, t), \tag{6a}$$
$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r}, t) = \mu_0 \mathbf{J}(\mathbf{r}, t). \tag{6b}$$

The associated Green function defined using the differential equation

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{7}$$

has solution

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{8}$$

This resembles the electric potential due to a unit point charge in electrostatics, however, it now accounts for dynamics, primarily as retardation in time.

(a) Show that the electric scalar potential and the magnetic vector potential, after completing the integral on t', are formally determined in terms of the following integrals,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho\left(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{9a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}\left(\mathbf{r}', t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}. \tag{9b}$$

(b) The non-retarded limit corresponds to the approximation

$$|\mathbf{r}-\mathbf{r}'| \ll c(t-t'). \tag{10}$$

Here $\mathbf{r}'$ and t', even though they are integral parameters, are physical, because they are associated to the distribution of sources. The non-retarded limit is consistent with assuming $c \to \infty$, the non-relativistic limit. Show that in this limit we have

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}, \tag{11a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{11b}$$

Though the resemblance is striking note that this is not still the static limit, because the time dependence in the magnetic vector potential contributes to the electric field.

(c) Radiation fields corresponds to the approximation

$$|\mathbf{r}-\mathbf{r}'| \gg c(t-t'). \tag{12}$$

Show that in this far-field limit

$$|\mathbf{r}-\mathbf{r}'| = r\left(1-\hat{\mathbf{r}}\cdot\frac{\mathbf{r}'}{r}\right) + \mathcal{O}\left(\frac{r'}{r}\right)^2. \tag{13}$$

Show that in the far-field approximation

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int d^3r'\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right), \tag{14a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right). \tag{14b}$$

(d) Define the retarded time

$$t_r = t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}. \tag{15}$$

Show that

$$\boldsymbol{\nabla} t_r = -\frac{\hat{\mathbf{r}}}{c} - \frac{1}{c}\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\frac{\mathbf{r}'}{r}\right) = -\frac{\hat{\mathbf{r}}}{c} + \mathcal{O}\left(\frac{r'}{r}\right). \tag{16}$$

(e) Show that the leading contributions are

$$\boldsymbol{\nabla}\phi(\mathbf{r},t) = -\frac{1}{c}\frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\right\}_{t'=t_r}, \tag{17a}$$

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}, \tag{17b}$$

$$\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t) = -\frac{1}{c}\frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}}{r}\times\int d^3r'\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}, \tag{17c}$$

Thus, derive

$$c\mathbf{B}(\mathbf{r},t) = -\hat{\mathbf{r}}\times\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}, \tag{18a}$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left[\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r} - c\hat{\mathbf{r}}\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\right\}_{t'=t_r}\right]. \tag{18b}$$

(f) Recall that Maxwell's equations implies the local charge conservation,

$$\frac{\partial}{\partial t'}\rho(\mathbf{r}',t') + \boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}',t') = 0. \tag{19}$$

Thus, we have

$$\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\right\}_{t'=t_r} = -\left\{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r=t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}}, \tag{20}$$

where we emphasize that the substitution $t' = t_r$ is made after completing the divergence with respect to $\mathbf{r}'$. By reversing this order show that

$$\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\right\}_{t'=t_r} = -\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}',t_r) + \frac{\hat{\mathbf{r}}}{c}\cdot\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}. \tag{21}$$

Under integration with respect to $\mathbf{r}'$ the first term on the right hand side contributes only on the surface. Thus, argue that this term does not contribute in the far-field zone. Then, recognize the vector triple product to deduce

$$\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{\mu_0}{4\pi} \frac{1}{r} \int d^3r' \left\{ \frac{\partial}{\partial t'} \mathbf{J}(\mathbf{r}',t') \right\}_{t'=t_r} \right). \tag{22}$$

(g) Verify that these fields satisfy

$$\mathbf{E} = -\hat{\mathbf{r}} \times c\mathbf{B}, \tag{23a}$$

$$c\mathbf{B} = \hat{\mathbf{r}} \times \mathbf{E}. \tag{23b}$$

Thus, $\hat{\mathbf{r}}$, $\mathbf{E}$, and $\mathbf{B}$, are orthogonal to each other. Further, we have $c^2B^2 = E^2$, which can be rewritten in the form

$$\frac{1}{2\mu_0} B^2 = \frac{1}{2} \varepsilon_0 E^2, \tag{24}$$

which states that the energy stored in the radiation field is equally divided in the electric and magnetic fields. Recall that plane monochromatic waves also satisfied these properties.