Solution to vibrations on a string ①

Date: 2019 Nov 14 for $H(x) = H_0 \sin^2\left(\pi \frac{x}{a}\right)$.

① $$\int_0^{2\pi} d\theta \, e^{in\theta} = \begin{cases} 2\pi, & \text{if } n=0, \\ \dfrac{\left(e^{in2\pi} - 1\right)}{in}, & \text{if } n \neq 0 \end{cases}$$

$$= \begin{cases} 2\pi, & \text{if } n=0, \\ \dfrac{(1)^n - 1}{in}, & \text{if } n \neq 0. \end{cases}$$

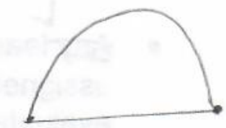
$$= 2\pi \, \delta_{n0}.$$

② $$\int_0^{\pi} d\theta \, e^{in\theta} = \begin{cases} \pi, & \text{if } n=0, \\ \dfrac{\left(e^{in\pi} - 1\right)}{in}, & \text{if } n \neq 0 \end{cases}$$

$$= \begin{cases} \pi, & \text{if } n=0, \\ \dfrac{(-1)^n - 1}{in}, & \text{if } n=0 \end{cases}$$

$$= \begin{cases} \pi, & \text{if } n=0, \\ 0, & \text{if } n = \pm 2, \pm 4, \cdots, \\ -\dfrac{2}{in}, & \text{if } n = \pm 1, \pm 3, \cdots \end{cases}$$

③ $$D_m = \frac{2}{a}\int_0^a dx \; H(x) \; \sin\left(m\pi \frac{x}{a}\right)$$



④ Let $H(x) = H_0 \sin^2\left(\pi \frac{x}{a}\right)$

$$D_m = \frac{2}{a}\int_0^a dx \; H_0 \sin^2\left(\pi \frac{x}{a}\right) \sin\left(m\pi \frac{x}{a}\right)$$

$$\pi \frac{x}{a} = \theta \qquad dx = \frac{a}{\pi} d\theta$$

$$D_m = \frac{2}{\cancel{a}} H_0 \int_0^\pi \frac{\cancel{a}}{\pi} d\theta \; \sin^2\theta \; \sin m\theta$$

$$= H_0 \frac{2}{\pi} \int_0^\pi d\theta \; \sin^2\theta \; \sin m\theta$$

⑤ $$D_m = H_0 \frac{2}{\pi} \int_0^\pi d\theta \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 \left(\frac{e^{im\theta} - e^{-im\theta}}{2i}\right)$$

$$= H_0 \frac{2}{\pi} \frac{1}{(-8i)} \int_0^\pi d\theta \left(e^{i2\theta} - 2 + e^{-i2\theta}\right)\left(e^{im\theta} - e^{-im\theta}\right)$$

$$= -\frac{H_0}{4\pi i} \int_0^\pi d\theta \Big[e^{i\theta(2+m)} - e^{i\theta(2-m)} - 2e^{im\theta} + 2e^{-im\theta} + e^{i\theta(m-2)} - e^{-i(2+m)\theta} \Big]$$

⑥ At $m=2$ two of the integrals have special values. Otherwise, they are given using standard integrals.

(7) $\underline{m = 2}$

$$D_2 = -\frac{H_0}{4\pi i}\left[\frac{(-1)^{2+m}-1}{i(2+m)} - \pi - 2\,\frac{(-1)^m-1}{im} + 2\,\frac{(-1)^m-1}{-im} + \pi - \frac{(-1)^{2+m}-1}{-i(2+m)}\right]_{m=2}$$

$$= -\frac{H_0}{4\pi i}\left[0 - \pi - 0 + 0 + \pi - 0\right]$$

$$= 0$$

(8) $m = 1, 3, 4, 5, \ldots$

$$D_m = -\frac{H_0}{4\pi i}\left[\frac{(-1)^{2+m}-1}{i(2+m)} - \frac{(-1)^{2-m}-1}{i(2-m)} - 2\,\frac{(-1)^m-1}{im} + 2\,\frac{(-1)^m-1}{-im} + \frac{(-1)^{m-2}-1}{i(m-2)} - \frac{(-1)^{2+m}-1}{-i(2+m)}\right]$$

$$= \frac{H_0}{4\pi}\left[\frac{(-1)^m-1}{(2+m)} - \frac{(-1)^m-1}{(2-m)} - 4\,\frac{(-1)^m-1}{m} + \frac{(-1)^m-1}{(2+m)} - \frac{(-1)^m-1}{(2-m)}\right]$$

$$= \frac{H_0}{4\pi}\left[(-1)^m-1\right]\left[\frac{2}{(2+m)} - \frac{2}{(2-m)} - \frac{4}{m}\right]$$

$$= \frac{H_0}{2\pi}\left[(-1)^m-1\right]\left[\frac{1}{(2+m)} - \frac{1}{(2-m)} - \frac{2}{m}\right]$$

$$= \frac{H_0}{2\pi}\left[(-1)^m-1\right]\left[\frac{1}{m+2} + \frac{1}{m-2} - \frac{2}{m}\right]$$

$$= \frac{H_0}{2\pi}\left[(-1)^m-1\right]\left[\frac{m(m-2) + m(m+2) - 2(m^2-4)}{m(m^2-4)}\right] = \frac{H_0}{2\pi}\left[(-1)^m-1\right]\frac{8}{m(m^2-4)}$$

(9) $m = 2, 4, 6, \dots$ (even)

$$D_m = 0$$

$m = 1, 3, 5, 7, \dots$ (odd)

$$D_m = \frac{H_0}{2\pi}(-2)\frac{8}{m(m^2-4)} = -\frac{H_0}{\pi}\frac{8}{m(m^2-4)}$$

(10) Solution:

$$h(x,t) = \sum_{m=1}^{\infty} D_m \cos\left(m\pi \frac{v}{a} t\right) \sin\left(m\pi \frac{x}{a}\right)$$

let $m = 2n+1$ $\quad n = 0, 1, 2, \dots$

$$= \sum_{n=0}^{\infty} -\frac{H_0}{\pi}\frac{8}{(2n+1)\left[(2n+1)^2-4\right]} \cos\left((2n+1)\pi \frac{v}{a} t\right) \sin\left((2n+1)\pi \frac{x}{a}\right)$$