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Orthogonality

① To prove

$$\frac{2}{a}\int_0^a dx\, \sin\left(m\pi\frac{x}{a}\right)\sin\left(m'\pi\frac{x}{a}\right) = \delta_{mm'}$$

m and m' are positive (non-zero) integers.

② Let $\pi\frac{x}{a} = \theta$ $\qquad dx = \frac{a}{\pi}\,d\theta$

③ $$I_{mm'} = \frac{2}{a}\int_0^a dx\, \sin\left(m\pi\frac{x}{a}\right)\sin\left(m'\pi\frac{x}{a}\right)$$

$$= \frac{2}{a}\int_0^{\pi}\frac{a}{\pi}\,d\theta\, \sin m\theta\, \sin m'\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} d\theta \left(\frac{e^{im\theta}-e^{-im\theta}}{2i}\right)\left(\frac{e^{im'\theta}-e^{-im'\theta}}{2i}\right)$$

$$= -\frac{1}{2\pi}\int_0^{\pi} d\theta\left[e^{i(m+m')\theta} - e^{i(m-m')\theta} - e^{-i(m-m')\theta} + e^{-i(m+m')\theta}\right]$$

④ Note that

$$\int_0^{\pi} d\theta\, e^{in\theta} = \begin{cases} \pi, & \text{if } n=0, \\ \dfrac{e^{in\pi}-1}{in}, & \text{if } n\neq 0. \end{cases}$$

5 $\underline{\underline{m = m'}}$

$$I_{mm} = -\frac{1}{2\pi}\left[\frac{\left(e^{i2m\pi}-1\right)}{i2m} - \pi - \pi + \frac{\left(e^{-i2m\pi}-1\right)}{-i2m}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{(1)^m - 1}{2mi} - 2\pi + \frac{(1)^m - 1}{-2mi}\right]$$

$$= 1$$

6 $m \neq m'$ (recall m and m' are positive integer)

$$I_{mm'} = -\frac{1}{2\pi}\left[\frac{\left(e^{i(m+m')\pi}-1\right)}{i(m+m')} - \frac{\left(e^{i(m-m')\pi}-1\right)}{i(m-m')} - \frac{\left(e^{-i(m-m')\pi}-1\right)}{-i(m-m')} + \frac{\left(e^{-i(m+m')\pi}-1\right)}{-i(m+m')}\right]$$

$$= -\frac{1}{2\pi i}\left[\frac{(-1)^{m+m'}-1}{(m+m')} - \frac{(-1)^{m-m'}-1}{(m-m')} + \frac{(-1)^{m-m'}-1}{(m-m')} - \frac{(-1)^{m+m'}-1}{(m+m')}\right]$$

$$= 0$$

7 Using 5 and 6 we have the orthogonality relation

$$\frac{2}{a}\int_0^a dx \, \sin\left(m\pi\frac{x}{a}\right)\sin\left(m'\pi\frac{x}{a}\right) = \delta_{mm'}$$