Final Exam (Fall 2019)

PHYS 301: Theoretical Methods in Physics

Date: 2019 Dec 13

Note: Standard identities will be provided to a student when requested.

1. (**20 points.**) Find the four roots of 1 by solving the equation

$$z^4 = 1. \tag{1}$$

Mark the the points corresponding to the four roots on the complex plane.

2. (**20 points.**) Find the eigenvalues and eigenvectors of the matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

3. (**20 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol in three dimensions evaluate the following using index notation.

$$\delta_{ij}\varepsilon_{ijk} = \tag{3a}$$

4. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla} \cdot (r^2 \mathbf{r}) = a\, r^n. \tag{4}$$

Thus, find a and n.

5. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} dx\, \delta(x-b) \frac{1}{\sqrt{x^2+a^2}}. \tag{5}$$

Given b is real.

6. (**20 points.**) The Legendre polynomials of order l, $-1 \leq x \leq 1$, are

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{6}$$

In particular,

$$P_0(x) = 1, \tag{7a}$$

$$P_1(x) = x, \tag{7b}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{7c}$$

Express the function

$$\sigma(\theta) = \cos^2 \theta \tag{8}$$

in terms of Legendre polynomials $P_l(\cos\theta)$. That is, determine the coefficients a_l in the series representation

$$\sigma(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta). \tag{9}$$

7. (**20 points.**) The Legendre polynomials of order l satisfy the recurrence relation

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x), \qquad l = 1, 2, 3, \ldots. \tag{10}$$

Given,

$$P_0(x) = 1, \tag{11a}$$
$$P_1(x) = x. \tag{11b}$$

Derive the explicit expression for $P_3(x)$.