Homework No. 04 (2019 Fall)

PHYS 301: Theoretical Methods in Physics

Due date: Friday, 2019 Sep 13, 10:00 AM, in class

1. Keywords: Eigenvalues and eigenvectors of a matrix; Matrix diagonalization; Properties of Pauli matrices; Eigenbasis dependence of matrices.

2. (**40 points.**) Consider the rotation matrix

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{1}$$

   (a) Find the eigenvalues of the matrix $\mathbf{A}$.

   (b) Find the normalized eigenvectors of matrix $\mathbf{A}$.

   (c) Determine the matrix that diagonalizes the matrix $\mathbf{A}$.

   (d) What can you then conclude about the eigenvalues and eigenvectors of $\mathbf{A}^{107}$? Find them.

3. (**45 points.**) A particular representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

   (In particular, these are Pauli matrices in the eigenbasis of σ_z.) Find the eigenvalues, normalized eigenvectors, and diagonalizing matrix, for each of the three Pauli matrix. Verify that your results satisfy the eigenvalue equation.

4. (**20 points.**) Construct the matrix

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}, \tag{3}$$

   where

$$\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{i}} + \sigma_y \hat{\mathbf{j}} + \sigma_z \hat{\mathbf{k}}, \tag{4}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}. \tag{5}$$

   Use the representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

   Find the eigenvalues of the matrix $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$.

5. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

Write σ_x in the eigenbasis of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{9}$$

Note that this representation has the arbitraryness of the choice of phase in the eigenvectors.