Homework No. 07 (2019 Fall)

PHYS 301: Theoretical Methods in Physics

Due date: Monday, 2019 Oct 14, 10:00 AM, in class

0. Keywords: Stern-Gerlach experiment, Heisenberg uncertainty relation, Commutation relations for operators, Matrix representation of an operator in the eigenbasis of another operator, orthogonality relations, completness relation.

1. (**20 points.**) Heisenberg uncertainty relations between position vector $\mathbf{x}$ and momentum vector $\mathbf{p}$ is the central postulate of quantum mechanics. It is the statement of non-commutativity and $\mathbf{x}$ and $\mathbf{p}$,

$$\mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \big[\mathbf{x}, \mathbf{p}\big] = i\hbar\mathbf{1}. \tag{1}$$

We also have

$$\big[\mathbf{x}, \mathbf{x}\big] = 0 \quad \text{and} \quad \big[\mathbf{p}, \mathbf{p}\big] = 0. \tag{2}$$

For angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, show that these relations imply

$$\big[L_x, L_y\big] = i\hbar L_z. \tag{3}$$

The commutation relations for angular momentum are summarized using the cyclic property of Levi-Civita symbol as $\big[L_i, L_j\big] = i\hbar\varepsilon_{ijk}L_k$.

2. (**20 points.**) Construct the matrix

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}, \tag{4}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{i}} + \sigma_y\hat{\mathbf{j}} + \sigma_z\hat{\mathbf{k}}, \tag{5}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}. \tag{6}$$

Use the representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Find the eigenvalues of the matrix $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$.

3. (**20 points.**) Show that

$$|+;\theta\rangle\langle+;\theta| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}. \tag{8}$$

4. (**20 points.**) The probabilities in a series setup of Stern-Gerlach experiment can be described using the notation

$$p([A = a'] \to [B = b'] \to [C = c']), \tag{9}$$

where $[A = a']$ denotes the selection of the beam corresponding to the eigenvalue a'. For spin-$\frac{1}{2}$ the operators A, B, and C, are given by $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the direction of the magnetic field. Connecting with the notation introduced in class, we have, for example, $[\sigma_x = +1] = [+; \theta = \frac{\pi}{2}, \phi = 0]$. Verify the following probabilities:

$$p([\sigma_x = +1] \to [\sigma_x = +1]) = 1, \tag{10a}$$

$$p([\sigma_x = +1] \to [\sigma_y = +1] \to [\sigma_x = +1]) = \frac{1}{4}, \tag{10b}$$

$$p([\sigma_x = +1] \to [\sigma_y = +1] \to [\sigma_x = -1]) = \frac{1}{4}, \tag{10c}$$

$$p([\sigma_x = +1] \to [\sigma_y = -1] \to [\sigma_x = +1]) = \frac{1}{4}, \tag{10d}$$

$$p([\sigma_x = +1] \to [\sigma_y = -1] \to [\sigma_x = -1]) = \frac{1}{4}. \tag{10e}$$

Does the measurement of σ_y *completely* wipe out the prior knowledge of the measurement of σ_x? If yes, why? If no, why not? (Hint: Argue that the measurement of σ_y has completely randomized the information content of σ_x, and thus wiped out the prior knowledge of the measurement of σ_x.) Are σ_x and σ_y complementary?