Homework No. 07 (2019 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2019 Oct 2, 2:00 PM, in class

0. Problems 1, 3, and 6, are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) The electric field due to a point dipole $\mathbf{d}$ at a distance $\mathbf{r}$ away from dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}\big]. \tag{1}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{d} = d\,\hat{\mathbf{z}}$.

   (a) Qualitatively plot the electric field lines for the dipole $\mathbf{d}$. (Hint: You do not have to depend on Eq. (1) for this purpose. An intuitive knowledge of electric field lines should be the guide.)

   (b) Find the (simplified) expression for the electric field on the positive z-axis. (Hint: On the positive z-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = z$.)

   (c) Find the (simplified) expression for the electric field on the negative z-axis.

2. (**20 points.**) The electric field of a point dipole $\mathbf{p}$ is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\Big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\Big] = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2}{r^5}\right]. \tag{2}$$

Evaluate $\boldsymbol{\nabla}\cdot\mathbf{E}$.
Hint: Intuitively, the divergence of a vector field is a measure of the density of source/sink of the field. For reference, the electric field lines drawn in Fig. 1 will be those of a point dipole in the limit of distance between the two charges going to zero, keeping the magnitude of the dipole moment fixed.

3. (**20 points.**) The force and torque on an electric dipole $\mathbf{d}$ in the presence of an electric field is given by

$$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E} \qquad \text{and} \qquad \boldsymbol{\tau} = \mathbf{d}\times\mathbf{E}, \tag{3}$$

respectively. Thus, describe the motion of an electric dipole when placed in between the plates of a parallel plate capacitor. Assume the plates to be perfectly conducting, of infinite cross-sectional area, and the medium in between to be vacuum.

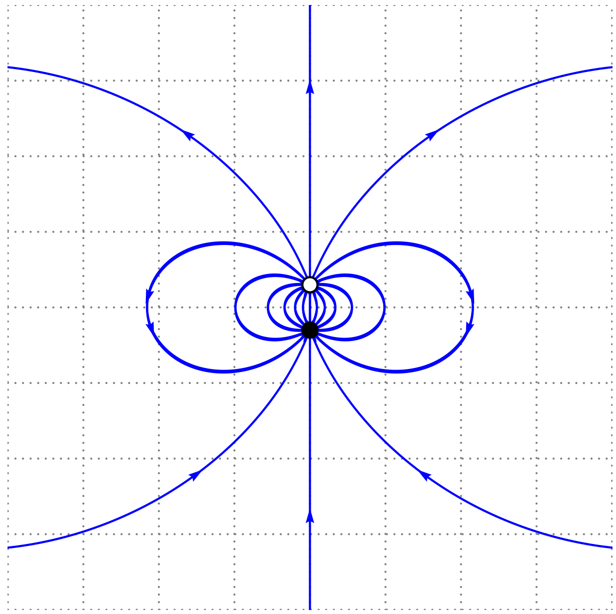

Figure 1: Electric field lines for a point dipole.

4. (**10 points.**) Interaction energy of a dipole $\mathbf{d}$ with an electric field $\mathbf{E}$ is

$$U = -\mathbf{d} \cdot \mathbf{E} = -dE\cos\theta. \tag{4}$$

The torque on the dipole due to the electric field is

$$\boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}. \tag{5}$$

Force is a manifestation of the systems tendency to minimize its energy, and in this spirit torque is defined as,

$$\tau = -\frac{\partial}{\partial\theta} U = -dE\sin\theta. \tag{6}$$

Show that there is no inconsistency, in sign, between the two definitions of torque.

5. (**20 points.**) Consider an infinitely thin flat sheet, of infinite extent, constructed out of a continuous distribution of point dipoles, all of them poiting in the direction of $\hat{\mathbf{z}}$, each of individual charge density

$$-\mathbf{p} \cdot \boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}_p), \tag{7}$$

where $\mathbf{r}_p$ is the position of an individual point dipole. The charge density of such a sheet is given by

$$\rho(\mathbf{r}) = -\sigma \frac{\partial}{\partial z}\delta(z), \tag{8}$$

where $\sigma = p\,\delta(x)\delta(y)$ is the electric dipole moment per unit area.

(a) Evaluate the electric potential for the sheet using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{9}$$

(Hint: Use the δ-function property to evaluate the z'-integral, after integrating by parts. The x' and y' integrals can be completed using standard substitutions.)

(b) Evaluate the electric field for the sheet by finding the gradient of the electric potential you calculated using Eq. (9),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{10}$$

6. (**20 points.**) Here is problem 4.31 of Griffiths 4th edition, which is not there in the 3rd edition:

> A point charge Q is "nailed down" on a table. Around it, at radius R is a frictionless circular track on which a dipole $\mathbf{d}$ rides, constrained always to point tangent to the circle. Use Eq. (4.5) of Griffiths, 4th/3rd edition, to show that the electric force on the dipole is
>
> $$\mathbf{F} = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}}{R^3}.$$
>
> Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

Footnote 21, in Griffiths 4th edition, is an acknowledgement: "This charming paradox was suggested by K. Brownstein."
You might also refer to comments by Prof. Alan Guth, in his Fall 2014 lecture notes, at
`http://web.mit.edu/8.07/www/probsets/ps06-f14.pdf`
`http://web.mit.edu/8.07/www/probsets/sol06-f14.pdf`

(a) The electric field of a point charge Q at distance $\mathbf{R}$ from the charge is

$$\mathbf{E}(\mathbf{R}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{R}}{R^3}. \tag{11}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d}\cdot\mathbf{E}. \tag{12}$$

Thus, derive the interaction energy between the charge Q and the dipole $\mathbf{d}$ to be

$$U = -\frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{R}}{R^3}. \tag{13}$$

(b) The variables in the problem are the coordinate ϕ that specifies the position of the dipole on the circular track, and the angle θ that the direction of the dipole makes with respect to the radius vector $\mathbf{R}$. Thus, conclude that the interaction energy is independent of the coordinate ϕ,

$$U(\theta) = -\frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{R^2}. \tag{14}$$

(c) The generalized tangential force on the dipole, upto a factor R, is

$$F_\phi = -\frac{\partial}{\partial\phi}U. \tag{15}$$

Thus, conclude that there is no tangential force acting on the dipole. No perpetual motion!

(d) The torque acting on the dipole is

$$F_\theta = -\frac{\partial}{\partial\theta}U. \tag{16}$$

Determine the angles for which this force in zero. Analyse each of these angles and find out if they are stable or unstable.

(e) Describe the motion of the dipole on the track for arbitrary initial conditions with respect to ϕ and θ. That is, describe your results in 6d.

7. (**30 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.9.)

(a) The electric field of a point charge q at distance $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}. \tag{17}$$

The force on a point dipole in the presence of an electric field is

$$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{18}$$

Use these to find the force on a point dipole due to a point charge.

(b) The electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{19}$$

The force on a point charge in the presence of an electric field is

$$\mathbf{F} = q\mathbf{E}. \tag{20}$$

Use these to find the force on a point charge due to a point dipole.

(c) Confirm that above two forces are equal in magnitude and opposite in direction, as per Newton's third law.

8. (**40 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.8.)
We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{21}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d}\cdot\mathbf{E}. \tag{22}$$

Further, the force between the two dipoles is given by

$$\mathbf{F} = -\boldsymbol{\nabla} U. \tag{23}$$

Use these expressions to derive

(a) the interaction energy between two point dipoles separated by distance $\mathbf{r}$ to be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[\mathbf{d}_1\cdot\mathbf{d}_2 - 3\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\big]. \tag{24}$$

(b) the force between the two dipoles to be

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}\big[(\mathbf{d}_1\cdot\mathbf{d}_2)\,\hat{\mathbf{r}} + (\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + (\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 - 5\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\big]. \tag{25}$$

(c) Are the forces central? That is, is the force in the direction of $\mathbf{r}$?

(d) Are the forces on the dipole equal in magnitude and opposite in direction? That is, do they satisfy Newton's third law?