Homework No. 08 (2019 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2019 Oct 9, 2:00 PM, in class

0. Problems 1, 4, 6, and 8, are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) Two electrons and two protons are placed at the corners of a square of side a, such that the electrons are at diagonally opposite corners.

   (a) What is the electric potential at the center of square?

   (b) What is the electric potential at the midpoint of either one of the sides of the square?

   (c) How much potential energy is required to move another proton from infinity to the center of the square?

   (d) How much additional potential energy is required to move the proton from the center of the square to one of the midpoint of either one of the sides of the square?

2. (**20 points.**) (Griffiths 4th edition, Problem 2.32) Two positive charges, q_1 and q_2 (masses m_1 and m_2) are at rest, held together by a massless string of length a. Now the string is cut, and the particles fly off in opposite directions. How fast is each one going, when they are far apart?

3. (**20 points.**) Consider two concentric spherical (perfectly) conducting shells, of radii a and $b > a$. The inner shell has a charge $+Q$ and the outer shell has a charge $-Q$.

   (a) Determine the expression for electric field everywhere.

   (b) Plot the magnitude of electric field as a function of the distance from the center of the concentric shells.

   (c) What is force experienced by another charge $+q$ a distance r from the center?

   (d) Plot the electric potential as a function of distance, choosing the the potential at the center to be zero.

4. (**20 points.**) The electric potential due to an infinitely thin plate (or a large disc of radius R on the xy-plane with $|x|, |y|, |z| \ll R$) with uniform charge density σ is given by the expression

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\big[R - |z|\big]. \tag{1}$$

   Find the (simplified) expression for the electric field due to the plane by evaluating the gradient of the above electric potential,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{2}$$

5. (**20 points.**) In class we evaluated the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involved the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{3}$$

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

6. (**20 points.**) The surface charge densities on the surface of two separate and independent charged spheres are given by

$$\sigma_1(\theta, \phi) = \frac{Q}{4\pi a^2}\cos\theta, \tag{4}$$

$$\sigma_2(\theta, \phi) = \frac{Q}{4\pi a^2}\cos^2\theta, \tag{5}$$

where θ is the polar angle in spherical coordinates. Calculate the total charge on each sphere by integrating over the surface of each sphere.

7. (**20 points.**) The charge density for a point charge q_a is described by

$$\rho(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a), \tag{6}$$

where $\mathbf{r}_a$ is the position of the charge.

(a) Evaluate the electric potential due to the point charge using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{7}$$

(Hint: Use the δ-function property to evaluate the integrals.)

(b) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (7),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{8}$$

(c) Evaluate the force exerted by the charge q_a on another charge q_b, at position $\mathbf{r}_b$, using the expression for electric field you obtained using Eq. (8) in

$$\mathbf{F} = q_b \mathbf{E}(\mathbf{r}_b). \tag{9}$$

To provide a check for your calculation, the answer for the expression for the force is provided here:

$$\mathbf{F} = \frac{q_a q_b}{4\pi\varepsilon_0}\frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3}. \tag{10}$$

8. (**20 points.**) The charge density for a perfectly conducting sphere of radius R with total charge Q on it is described by

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\delta(r - R). \tag{11}$$

   (a) Evaluate the integral

$$\int d^3r' \rho(\mathbf{r}') \tag{12}$$

   over all space.

   (b) Evaluate the electric potential of the sphere inside and outside the sphere using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{13}$$

   (Hint: Use the δ-function property to evaluate the r' integral. Choose the observation pont to be on the z axis, which allows the θ' and ϕ' integrals to be evaluated.)

   (c) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (13),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{14}$$

9. (**20 points.**) Consider a line segment of length $2L$ with uniform line charge density λ.

   (a) When the rod is placed on the z-axis centered on the origin, show that the charge density can be expressed as

$$\rho(\mathbf{r}) = \lambda\delta(x)\delta(y)\theta(-L < z < L), \tag{15}$$

   where $\theta(-L < z < L) = 1$, if $-L < z < L$ and $\theta(z) = 0$, otherwise.

   (b) Inverting the Poisson equation for the electric potential, using the Green's function, evaluate the electric potential for the rod as

$$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L - z}{\sqrt{x^2 + y^2}}\right) + \sinh^{-1}\left(\frac{L + z}{\sqrt{x^2 + y^2}}\right)\right]. \tag{16}$$

   (c) Using $\sinh t = (e^t - e^{-t})/2$, show that

$$\sinh^{-1} t = \ln(t + \sqrt{t^2 + 1}). \tag{17}$$

   (d) Thus, express the electric potential of Eq. (16) in the form

$$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L}, \frac{\rho}{L}\right)\right], \tag{18}$$

   where $\rho^2 = x^2 + y^2$ and

$$F(a, b) = \ln[1 - a + \sqrt{(1 - a)^2 + b^2}] + \ln[1 + a + \sqrt{(1 + a)^2 + b^2}]. \tag{19}$$

(e) An infinite rod (on the z axis) is obtained by taking the limit $\rho \ll L, z \ll L$. Show that

$$\phi(\mathbf{r}) \xrightarrow{\rho \ll L, z \ll L} -\frac{2\lambda}{4\pi\varepsilon_0} \ln \frac{\rho}{2L}. \tag{20}$$

Hint: Series expand and keep only leading order terms.

(f) Using $\mathbf{E} = -\boldsymbol{\nabla}\phi$ determine the electric field for an infinite rod (placed on the z-axis) to be

$$\mathbf{E}(\mathbf{r}) = \frac{2\lambda}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\rho}}}{\rho}. \tag{21}$$