Homework No. 10 (2019 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2019 Oct 30, 2:00 PM, in class

0. Problems 1, 2, and 7 are to be submitted for assessment. Problem 10 is useful for conceptual understanding. Rest are for practice.

1. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
```
Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x],
LegendreP[4,x] },{x,-1,1}]
```
Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

2. (**20 points.**) Legendre polynomials are conveniently generated using the relation
$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{1}$$
where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

3. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation
$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{2}$$
Verify this explicitly for $l = 0, 1, 2, 3, 4$.

4. (**20 points.**) Legendre polynomials satisfy the differential equation
$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{3}$$
Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation
$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{4}$$
Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$.

6. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x'). \tag{5}$$

This is for your information. No work needed.

7. (**10 points.**) The surface charge density on the surface of a charged sphere is given by

$$\sigma(\theta, \phi) = \frac{Q}{4\pi a^2} \cos^2\theta, \tag{6}$$

where θ is the polar angle in spherical coordinates. Express this charge distribution in terms of the Legendre polynomials. Recall,

$$P_0(\cos\theta) = 1, \tag{7a}$$
$$P_1(\cos\theta) = \cos\theta, \tag{7b}$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}. \tag{7c}$$

8. (**20 points.**) Consider the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involved the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{8}$$

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

9. (**10 points.**) The induced charge on the surface of a spherical conducting shell of radius a due to a point charge q placed a distance b away from the center is given by

$$\rho(\mathbf{r}) = \sigma(\theta, \phi)\, \delta(r - a), \tag{9}$$

where

$$\sigma(\theta, \phi) = -\frac{q}{4\pi a} \frac{(r_>^2 - r_<^2)}{(a^2 + b^2 - 2ab\cos\theta)^{\frac{3}{2}}}, \tag{10}$$

where $r_< = \mathrm{Min}(a, b)$ and $r_> = \mathrm{Max}(a, b)$. Calculate the dipole moment of this charge configuration (excluding the original charge q) using

$$\mathbf{d} = \int d^3r\, \mathbf{r}\, \rho(\mathbf{r}), \tag{11}$$

for the two cases $a < b$ and $a > b$, representing the charge being inside or outside the sphere. (Hint: First complete the r integral and the ϕ integral. Then, for the θ integral substitute $a^2 + b^2 - 2ab\cos\theta = y$.)

10. (**40 points.**) Recollect Legendre polynomials

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{12}$$

In particular

$$\begin{aligned} P_0(x) &= 1, && (13a)\\ P_1(x) &= x, && (13b)\\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}. && (13c) \end{aligned}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r'-a). \tag{14}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned} \phi(r,0,0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}}, \end{aligned} \tag{15}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_0(\cos\theta). \tag{16}$$

Evaluate the integral in Eq. (15) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{17}$$

where $r_< = \text{Min}(a,r)$ and $r_> = \text{Max}(a,r)$.
Note: This was done in class. Nevertheless, present the relevant steps.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{18}$$

Evaluate the integral in Eq. (15) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{3}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right). \tag{19}$$

Note: This was done in class. Nevertheless, present the relevant steps.

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{20}$$

Evaluate the integral in Eq. (15) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left(\frac{r_<}{r_>} \right)^2 . \tag{21}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{22}$$

the integral in Eq. (15) leads to

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(2l+1)} \frac{1}{r_>} \left(\frac{r_<}{r_>} \right)^l . \tag{23}$$