

Homework No. 12 (2019 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2019 Nov 20, 2:00 PM, in class

0. Keywords: Electric polarization, dielectric function
0. Problems 1, 2, 4, and 7 are to be submitted for assessment. Rest are for practice.
1. (**Example.**) Consider a uniformly polarized slab of thickness a , that has the direction of its electric polarization (electric dipole moment density) in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma \hat{\mathbf{z}} \left[\theta(z) - \theta(z - a) \right] = \begin{cases} 0, & z < 0, \\ \sigma \hat{\mathbf{z}}, & 0 < z < a, \\ 0, & a < z, \end{cases} \quad (1)$$

where σ characterizes the polarization of the slab. Note that σ is dipole moment per unit volume, which has dimensions of charge per unit area.

- (a) Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\nabla \cdot \mathbf{P} \quad (2)$$

and show that

$$\rho_{\text{eff}}(\mathbf{r}) = -\sigma\delta(z) + \sigma\delta(z - a). \quad (3)$$

Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density \mathbf{P} leads to a surface charge density.

- (b) Find the total charge in the slab using

$$Q_{\text{en}} = \int d^3r \rho_{\text{eff}}(\mathbf{r}). \quad (4)$$

2. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma \hat{\mathbf{x}} \theta(-z), \quad (5)$$

where σ is the polarization per unit area of the slab. Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}. \quad (6)$$

3. (**Example.**) Consider a solid sphere of radius R with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0 \theta(R - r), \quad (7)$$

where \mathbf{P}_0 is a uniform vector, $\theta(x)$ is the Heaviside step function, and $r^2 = x^2 + y^2 + z^2$.

- (a) Show that the effective charge density due to the polarization is

$$\rho_{\text{eff}}(\mathbf{r}) = -\nabla \cdot \mathbf{P} = (\mathbf{P}_0 \cdot \hat{\mathbf{r}}) \delta(r - R). \quad (8)$$

- (b) If we choose polarization to be along the direction of $\hat{\mathbf{z}}$, that is,

$$\mathbf{P}_0 = \sigma \hat{\mathbf{z}}, \quad (9)$$

we have, using $\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \cos \theta$,

$$\rho_{\text{eff}}(\mathbf{r}) = \sigma \cos \theta \delta(r - R). \quad (10)$$

Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density \mathbf{P} leads to a surface charge density.

- (c) Find the enclosed charge inside an arbitrary sphere of radius r using

$$Q_{\text{en}} = \int d^3r \rho_{\text{eff}}(\mathbf{r}) \quad (11)$$

for $r < R$ and $r > R$.

4. (**20 points.**) Consider a uniformly polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha \mathbf{r} \theta(R - r). \quad (12)$$

- (a) Calculate $-\nabla \cdot \mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R - r) + \alpha r\delta(r - R). \quad (13)$$

- (b) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r' \rho_{\text{eff}}(\mathbf{r}') \quad (14)$$

for $r < R$ and $r > R$.

5. (**20 points.**) A permanently polarized sphere of radius R is described by the polarization vector

$$\mathbf{P}(\mathbf{r}) = \alpha r^2 \hat{\mathbf{r}} \theta(R - r). \quad (15)$$

Find the effective charge density by calculating $-\nabla \cdot \mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R - r)$ that is interpreted as a volume charge density, and another containing $\delta(R - r)$ that can be interpreted as a surface charge density.

6. (**Example.**) Consider a solid right circular cylinder of radius R with axis along the z axis and of infinite length with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0 \theta(R - \rho), \quad (16)$$

where $\rho^2 = x^2 + y^2$ and \mathbf{P}_0 is perpendicular to the axis of the cylinder. We shall find the electric potential and the electric field outside the cylinder.

- (a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\nabla \cdot \mathbf{P} = \mathbf{P}_0 \cdot \hat{\rho} \delta(\rho - R). \quad (17)$$

- (b) Discuss the case when \mathbf{P}_0 is parallel to the axis of the cylinder. Further, qualitatively, discuss the case if, in addition, the cylinder was of finite length in the direction of z .
7. (**Example.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma \hat{\mathbf{z}} \theta(-z), \quad (18)$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?

8. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma \hat{\mathbf{x}} \theta(-z), \quad (19)$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?