

Final Exam (Fall 2019)

PHYS 500A: Mathematical Methods

Date: 2019 Dec 12

1. (100 points.) The cylindrical Green's function satisfies

$$\left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \varepsilon(\rho) \rho \frac{\partial}{\partial \rho} + \varepsilon(\rho) \frac{m^2}{\rho^2} + \varepsilon(\rho) k_z^2 \right] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \quad (1)$$

Consider a dielectric cylinder described by

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2 & \text{for } \rho < a, \\ \varepsilon_1 & \text{for } a < \rho. \end{cases} \quad (2)$$

(a) Integrate Eq. (1) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \quad (3a)$$

$$\varepsilon(\rho) \rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \quad (3b)$$

(b) Integrate Eq. (1) around $\rho = a$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z) \Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0, \quad (4a)$$

$$\varepsilon(\rho) \rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z) \Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0. \quad (4b)$$

(c) For

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2, & \rho < a, \\ \varepsilon_1, & a < \rho, \end{cases} \quad (5)$$

derive the solution

$$\underline{\rho' < a:}$$

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) I_m(k_z \rho') \frac{K_a K'_a}{\Delta}, & \rho, \rho' < a, \\ \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) I_m(k_z \rho') \frac{I'_a K_a}{\Delta}, & \rho' < a < \rho. \end{cases} \quad (6)$$

$a < \rho'$:

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) K_m(k_z \rho') \frac{I_a K'_a}{\Delta}, & \rho < a < \rho', \\ \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) K_m(k_z \rho') \frac{I_a I'_a}{\Delta}, & a < \rho, \rho'. \end{cases} \quad (7)$$

We used the definitions

$$\frac{1}{\Delta} = \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 I_a K'_a - \varepsilon_2 K_a I'_a)}, \quad I_a \equiv I_m(k_z a), \quad K_a \equiv K_m(k_z a). \quad (8)$$

(d) Show that in the perfect conductor limit

Inside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) I_m(k_z \rho') \frac{K_a}{I_a}. \quad (9)$$

Outside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) K_m(k_z \rho') \frac{I_a}{K_a}. \quad (10)$$