Midterm Exam No. 02 (Fall 2019)

PHYS 500A: Mathematical Methods

Date: 2019 Oct 29

1. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx\, e^{iax}}{x^2+1} \tag{1}$$

using Cauchy's theorem, after choosing a suitable contour. Here a is real.

2. (**20 points.**) Consider the integral

$$I(\theta) = \frac{1}{\pi} \int_0^{\infty} \frac{x^{\frac{1}{2}}\, dx}{1 + 2x\cos\theta + x^2}, \tag{2}$$

where $0 \leq \theta < 2\pi$ is real. To evaluate $I(\theta)$ let us consider the following integral on the complex plane

$$G(\theta) = \frac{1}{\pi} \oint_c \frac{z^{\frac{1}{2}}\, dz}{1 + 2z\cos\theta + z^2}, \tag{3}$$

where the contour c is described in Figure 1.

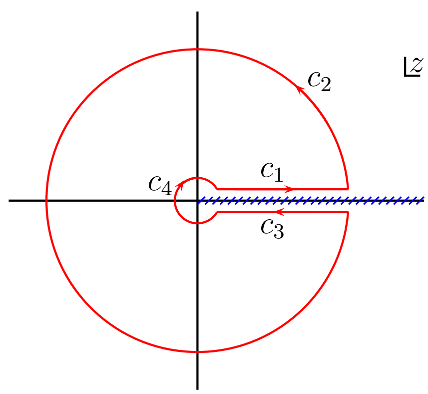

Figure 1: Contour $c = c_1 + c_2 + c_3 + c_4$. The radii of the contours c_2 and c_4 are R and ϵ, respectively, and contours c_1 and c_3 are δ away from the real line. We assume limits $\epsilon \to 0$, $R \to \infty$, and $\delta \to 0$.

(a) Show that

$$1 + 2z\cos\theta + z^2 = (z + e^{i\theta})(z + e^{-i\theta}) \tag{4}$$

and identify the poles. Show that the integrand has a branch point at $z = 0$. Choose the branch cut to be the positive real line. Using Cauchy's theorem show that

$$G(\theta) = 2\frac{\sin\frac{\theta}{2}}{\sin\theta}. \tag{5}$$

(b) Next, let us evaluate $G(\theta)$ by evaluating the integrals on the contour explicitly.

i. For the part of contour constituting c_1 substitute $z = xe^{i\delta} \sim x + i\delta'$ and show that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\lim_{R\to\infty}\frac{1}{\pi}\oint_{c_1}\frac{z^{\frac{1}{2}}\,dz}{(z + e^{i\theta})(z + e^{-i\theta})} = I(\theta). \tag{6}$$

ii. For the part of contour constituting c_3 substitute $z = xe^{i(2\pi-\delta)} \sim x - i\delta'$ and show that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\lim_{R\to\infty}\frac{1}{\pi}\oint_{c_3}\frac{z^{\frac{1}{2}}\,dz}{(z + e^{i\theta})(z + e^{-i\theta})} = -e^{i\frac{2\pi}{2}}I(\theta) = I(\theta). \tag{7}$$

iii. For the part of contour constituting c_2 substitute $z = Re^{i\theta}$ and show that

$$\lim_{R\to\infty}\frac{1}{\pi}\oint_{c_2}\frac{z^{\frac{1}{2}}\,dz}{(z + e^{i\theta})(z + e^{-i\theta})} = 0. \tag{8}$$

iv. For the part of contour constituting c_4 substitute $z = \epsilon e^{i\theta}$ and show that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\oint_{c_2}\frac{z^{\frac{1}{2}}\,dz}{(z + e^{i\theta})(z + e^{-i\theta})} = 0. \tag{9}$$

(c) Together, conclude that

$$2\frac{\sin\frac{\theta}{2}}{\sin\theta} = I(\theta) + 0 + I(\theta) + 0. \tag{10}$$

Thus, evaluate $I(\theta)$.

3. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0\frac{d}{dt} + \omega_0^2\right]x(t) = 0, \tag{11}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$. Plot $x(t)$ as a function of t in the graph in Figure 2, where ω_0 and v_0/ω_0 is used to set scales for time t and position $x(t)$. For what t is the solution $x(t)$ a maximum?

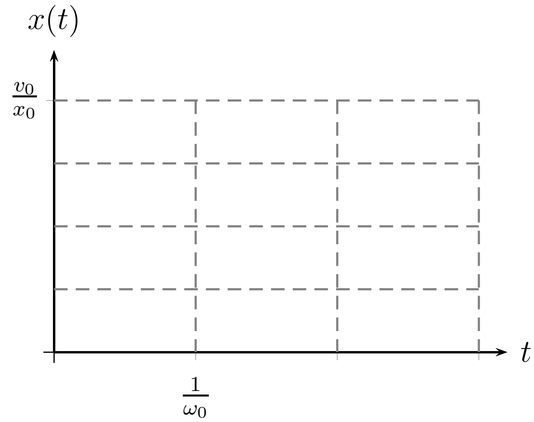

Figure 2: Critically damped harmonic oscillator.

4. (**20 points.**) Read the article titled 'Life at low Reynolds number' by E. M. Purcell, American Journal of Physics 45 (1977) 3. Here is the link to the article:

   http://dx.doi.org/10.1119/1.10903

   Here is a question asked to verify the understanding of the concept being discussed in the paper. Imagine a micrometer sized bacteria, shaped like a human, swimming in water using the methods used by a typical human swimmer. Qualitatively describe the motion of this hypothetical bacteria.