Homework No. 02 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2019 Sep 3, 4.00pm

1. (**50 points.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in cylindrical polar coordinates. In particular, we have

$$x^1 = x = \rho\cos\phi, \qquad u^1 = \rho = \sqrt{x^2+y^2}, \tag{1a}$$

$$x^2 = y = \rho\sin\phi, \qquad u^2 = \phi = \tan^{-1}\frac{y}{x}, \tag{1b}$$

$$x^3 = z = z, \qquad u^3 = z = z, \tag{1c}$$

Let us define the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$

$$\hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial\mathbf{r}}{\partial u^i}. \tag{3}$$

Show that

$$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}_2 = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}. \tag{4}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{5}$$

Show that

$$\mathbf{e}^1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \qquad \mathbf{e}^3 = \hat{\mathbf{z}}. \tag{6}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i\cdot\mathbf{e}_j = \delta^i_j. \tag{7}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i\hat{\mathbf{x}}_i = du^i\mathbf{e}_i. \tag{8}$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{9}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{10}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\mathbf{r} = \mathbf{1} \tag{11}$$

derive the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1}. \tag{12}$$

Express the completeness relation in cylindrical polar coordinates in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

(f) Transformation matrix: The components of a vector $\mathbf{A}$ are defined using the relations

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = A^i \hat{\mathbf{x}}_i = \bar{A}^i \mathbf{e}_i, \tag{13a}$$

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = \hat{\mathbf{x}}^i A_i = \mathbf{e}^i \bar{A}_i. \tag{13b}$$

Then, derive the transformations

$$\bar{A}^j = A^i T_i^{\ j}, \qquad T_i^{\ j} = \hat{\mathbf{x}}_i \cdot \mathbf{e}^j, \tag{14a}$$

$$\bar{A}_j = S_j^{\ i} A_i, \qquad S_j^{\ i} = \mathbf{e}_j \cdot \hat{\mathbf{x}}^i, \tag{14b}$$

and show that $T_i^{\ j} S_j^{\ k} = \delta_i^k$. Find S and T for cylindrical polar coordinates.

2. (**50 points.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in spherical polar coordinates. In particular, we have

$$x^1 = x = r\sin\theta\cos\phi, \qquad u^1 = r = \sqrt{x^2 + y^2 + z^2}, \tag{15a}$$

$$x^2 = y = r\sin\theta\sin\phi, \qquad u^2 = \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \tag{15b}$$

$$x^3 = z = r\cos\theta, \qquad u^3 = \phi = \tan^{-1}\frac{y}{x}, \tag{15c}$$

Let us define the unit vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{16a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{16b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{16c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{17}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{r}}, \qquad \mathbf{e}_2 = r\hat{\boldsymbol{\theta}}, \qquad \mathbf{e}_3 = r\sin\theta\hat{\boldsymbol{\phi}}. \tag{18}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{19}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{r}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\theta}}}{r}, \qquad \mathbf{e}^3 = \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}. \tag{20}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{21}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \tag{22}$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{23}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{24}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\mathbf{r} = \mathbf{1} \tag{25}$$

derive the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1}. \tag{26}$$

Express the completeness relation in spherical polar coordinates in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

(f) Transformation matrix: The components of a vector $\mathbf{A}$ are defined using the relations

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = A^i \hat{\mathbf{x}}_i = \bar{A}^i \mathbf{e}_i, \tag{27a}$$

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = \hat{\mathbf{x}}^i A_i = \mathbf{e}^i \bar{A}_i. \tag{27b}$$

Then, derive the transformations

$$\bar{A}^j = A^i T_i^{\ j}, \qquad T_i^{\ j} = \hat{\mathbf{x}}_i \cdot \mathbf{e}^j, \tag{28a}$$

$$\bar{A}_j = S_j^{\ i} A_i, \qquad S_j^{\ i} = \mathbf{e}_j \cdot \hat{\mathbf{x}}^i, \tag{28b}$$

and show that $T_i^{\ j} S_j^{\ k} = \delta_i^k$. Find S and T for spherical polar coordinates.