(Follow-up on) Homework No. 03 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2019 Sep 10, 4.00pm

1. (**20 points.**) Let (ρ, ϕ, z) represent cylindrical polar coordinates. For studying a phenomenon on a plane is it convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{1}$$

$$\boldsymbol{\nabla}_\rho = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}. \tag{2}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{3a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\, 2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{3b}$$

2. (**20 points.**) [This question has errors. It has been modified into Problem 3 in a follow-up.]

Let (r, θ, ϕ) represent spherical polar coordinates. For studying a phenomenon on the surface of a sphere it is convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp + \hat{\mathbf{r}}\frac{\partial}{\partial r}, \tag{4}$$

$$\boldsymbol{\nabla}_\perp = \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{5}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\perp \cdot \left(\frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\right) = 2\pi\Big[\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) - \delta^{(2)}(\mathbf{r}_\perp - \mathbf{S})\Big], \tag{6a}$$

$$\boldsymbol{\nabla}_\perp \times \left(\frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\right) = 0, \tag{6b}$$

and

$$\boldsymbol{\nabla}_\perp \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\right) = 0, \tag{7a}$$

$$\boldsymbol{\nabla}_\perp \times \left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\right) = \hat{\mathbf{r}}\, 2\pi\Big[\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) - \delta^{(2)}(\mathbf{r}_\perp - \mathbf{S})\Big], \tag{7b}$$

where $\mathbf{N}$ represents the North pole and $\mathbf{S}$ represents the South pole on the sphere. In particular,

$$\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) = \frac{\delta(\theta)\delta(\phi)}{r^2 \sin\theta}, \tag{8}$$

$$\delta^{(2)}(\mathbf{r}_\perp - \mathbf{S}) = \frac{\delta(\theta - \pi)\delta(\phi)}{r^2 \sin\theta}. \tag{9}$$

3. (**20 points.**) [Problem 2 has an error. The left hand side of Eq. (6b) does not evaluate to zero. Here is the modified version of Problem 2.]

   Let us consider the following fields that exist only the surface of a sphere of radius a:

$$\mathbf{E} = \hat{\boldsymbol{\theta}} \frac{\delta(r-a)}{2\pi r \sin\theta}, \tag{10a}$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\delta(r-a)}{2\pi r \sin\theta}, \tag{10b}$$

   where (r, θ, ϕ) are spherical polar coordinates.

   (a) Illustrate the vector field lines for $\mathbf{E}$ and $\mathbf{B}$ using a diagram.

   (b) Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \quad \theta \neq 0, \pi, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \text{everywhere}, \tag{11a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \quad \text{everywhere}, \qquad \boldsymbol{\nabla} \times \mathbf{B} = 0, \quad \theta \neq 0, \pi. \tag{11b}$$

   (c) Further, using Gauss's theorem and Stoke's theorem, show that

$$\int_V d^3r\, \boldsymbol{\nabla} \cdot \mathbf{E} = \begin{cases} +1, & \text{if } V \text{ encloses } \theta = 0, \\ -1, & \text{if } V \text{ encloses } \theta = \pi, \end{cases} \tag{12a}$$

$$\int_S d\mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \begin{cases} +1, & \text{if } S \text{ encloses } \theta = 0, \\ -1, & \text{if } S \text{ encloses } \theta = \pi, \end{cases} \tag{12b}$$

$$\tag{12c}$$

   where V represents the volume of a cone with apex at the origin with infinitely small opening angle, and S represents an infinitely small surface area on the sphere.

   Thus, using the property of δ-function, deduce

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \left[\delta^{(3)}(\mathbf{r} - \mathbf{N}) - \delta^{(3)}(\mathbf{r} - \mathbf{S})\right], \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{13a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \hat{\mathbf{r}} \left[\delta^{(3)}(\mathbf{r} - \mathbf{N}) - \delta^{(3)}(\mathbf{r} - \mathbf{S})\right], \tag{13b}$$

   where $\mathbf{N}$ represents the North pole and $\mathbf{S}$ represents the South pole on the sphere. In particular,

$$\delta^{(3)}(\mathbf{r} - \mathbf{N}) = \delta(r-a) \frac{\delta(\theta)\delta(\phi)}{r^2 \sin\theta}, \tag{14a}$$

$$\delta^{(3)}(\mathbf{r} - \mathbf{S}) = \delta(r-a) \frac{\delta(\theta - \pi)\delta(\phi)}{r^2 \sin\theta}. \tag{14b}$$