Homework No. 03 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2019 Sep 10, 4.00pm

1. (**20 points.**) Let (ρ, ϕ, z) represent cylindrical polar coordinates. For studying a phenomenon on a plane is it convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{1}$$

$$\boldsymbol{\nabla}_\rho = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}. \tag{2}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{3a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\, 2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{3b}$$

2. (**20 points.**) Let (r, θ, ϕ) represent spherical polar coordinates. For studying a phenomenon on the surface of a sphere it is convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp + \hat{\mathbf{r}}\frac{\partial}{\partial r}, \tag{4}$$

$$\boldsymbol{\nabla}_\perp = \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{5}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\perp \cdot \left(\frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\right) = 2\pi\Big[\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) - \delta^{(2)}(\mathbf{r}_\perp - \mathbf{S})\Big], \tag{6a}$$

$$\boldsymbol{\nabla}_\perp \times \left(\frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\right) = 0, \tag{6b}$$

and

$$\boldsymbol{\nabla}_\perp \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\right) = 0, \tag{7a}$$

$$\boldsymbol{\nabla}_\perp \times \left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\right) = \hat{\mathbf{r}}\, 2\pi\Big[\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) - \delta^{(2)}(\mathbf{r}_\perp - \mathbf{S})\Big], \tag{7b}$$

where **N** represents the North pole and **S** represents the South pole on the sphere. In particular,

$$\delta^{(2)}(\mathbf{r}_\perp - \mathbf{N}) = \frac{\delta(\theta)\delta(\phi)}{r^2 \sin\theta}, \tag{8}$$

$$\delta^{(2)}(\mathbf{r}_\perp - \mathbf{S}) = \frac{\delta(\theta - \pi)\delta(\phi)}{r^2 \sin\theta}. \tag{9}$$