Homework No. 04 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2019 Sep 17, 4.00pm

1. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann conditions. That is, the real and imaginary parts of an analytic function

$$f(x+iy) = u(x,y) + iv(x,y) \tag{1}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2b}$$

In terms of the variables z and z^*, defined using

$$z = x + iy, \qquad x = \frac{z+z^*}{2}, \tag{3a}$$

$$z^* = x - iy, \qquad y = \frac{z-z^*}{2i}, \tag{3b}$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}, \tag{4a}$$

$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial z^*}, \qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}. \tag{4b}$$

(a) Show that the conditions in Eqs. (2) imply

$$\frac{\partial f}{\partial z^*} = -\frac{\partial}{\partial z}(u - iv), \tag{5a}$$

$$\frac{\partial f}{\partial z^*} = +\frac{\partial}{\partial z}(u - iv), \tag{5b}$$

respectively. Thus, show that the conditions in Eqs. (2) imply

$$\frac{\partial f}{\partial z^*} = 0, \tag{6}$$

which is insightful.

(b) Is the inverse true? That is, does the condition in Eq. (6) imply the conditions in Eqs. (2). To this end, begin from Eq. (6) and immediately conclude

$$\frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*} = 0. \tag{7}$$

Then, proceed to derive

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \tag{8}$$

which implies the conditions in Eqs. (2).

2. (**20 points.**) Check if the function

$$f(z) = e^z + e^{iz} \tag{9}$$

satisfies the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

3. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \tag{10}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = -\frac{1}{z^2}, \qquad z \neq 0. \tag{11}$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

4. (**20 points.**) Plot

$$f(x) = e^{-\frac{1}{x}}, \tag{12}$$

given x is positive and real. Also, imagine the plot for $f(iy) = e^{-\frac{1}{iy}}$. Let z be complex. Check if the function

$$f(z) = e^{-\frac{1}{z}} \tag{13}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = \frac{e^{-\frac{1}{z}}}{z^2}, \qquad z \neq 0. \tag{14}$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

5. (**30 points.**) The close connection between the geometry of a complex number

$$z = x + iy \tag{15}$$

and a two-dimensional vector

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{16}$$

is intriguing. They have the same rules for addition and subtraction, but differ in their rules for multiplication. Show that

$$z_1^* z_2 = (\mathbf{r}_1 \cdot \mathbf{r}_2) + i(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{k}}. \tag{17}$$

In the quest for a number system that corresponds to a three dimensional vector, Hamilton in 1843 invented the quaternions. A quaternion P can be expressed in terms of Pauli matrices as

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{18}$$

Recall that the Pauli matrices are completely characterized by the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{19}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{20a}$$

$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{20b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{21}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.
Solution:

$$\big[P, Q\big] = -2i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{22}$$