Homework No. 06 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Thursday, 2019 Oct 4, 4.00pm

1. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z^2 - a^2)}, \tag{1}$$

where the contour c is a unit circle going counterclockwise with center at the origin. Inquire the cases when $|a| > 1$ and $|a| < 1$.

2. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx\, e^{iax}}{x^2 + 1} \tag{2}$$

using Cauchy's theorem, after choosing a suitable contour. Here a is real.

3. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{1 - 2a\cos\theta + a^2}, \tag{3}$$

where a is complex.

   (a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{4}$$

   and express the integral as a contour integral along the unit circle going counterclockwise. Locate the poles.

   (b) Evaluate the residues and show that

$$I(a) = \begin{cases} \dfrac{1}{1 - a^2}, & \text{if } |a| < 1, \\ \dfrac{1}{a^2 - 1}, & \text{if } |a| > 1. \end{cases} \tag{5}$$

   (c) Plot $I(a)$ for real values of a. Plot real and imaginary part of $I(a)$ for complex a. Argue that $I(1)$ is divergent.