

Homework No. 09 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2019 Nov 5, 4.00pm

1. (40 points.) Verify that

$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \quad (1)$$

where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that

$$\frac{d^2}{dz^2}|z| = 2\delta(z). \quad (2)$$

Also, argue that, for a well defined function $f(z)$, the replacement

$$f(z)\delta(z) = f(0)\delta(z) \quad (3)$$

is justified. Using Eq. (1), Eq. (2), and Eq. (3), verify (by substituting the solution into the differential equation) that

$$g(z) = \frac{1}{2k} e^{-k|z|} \quad (4)$$

is a particular solution of the differential equation

$$-\left(\frac{d^2}{dz^2} - k^2\right)g(z) = \delta(z). \quad (5)$$

2. (20 points.) Show that

$$\bar{\delta}(x) = -x \frac{d}{dx} \delta(x) \quad (6)$$

is also a model for the δ -function by showing that

$$\int_{-\infty}^{\infty} dx \bar{\delta}(x) f(x) = f(0). \quad (7)$$

Hint: Integrate by parts.

3. (60 points.) A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2} + \omega^2\right)x(t) = F(t), \quad (8)$$

where ω is the angular frequency of the oscillator and $F(t)$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2} + \omega^2\right)G(t, t') = \delta(t - t'). \quad (9)$$

The continuity conditions satisfied by the Green function are

$$\left.\frac{d}{dt}G(t, t')\right|_{t=t'-\delta}^{t=t'+\delta} = -1 \quad (10)$$

and

$$G(t, t')\Big|_{t=t'-\delta}^{t=t'+\delta} = 0. \quad (11)$$

(a) Verify that a particular solution,

$$G_R(t - t') = -\frac{1}{\omega}\theta(t - t') \sin \omega(t - t'), \quad (12)$$

which is referred to as the retarded Green's function, satisfies the Green function differential equation and the continuity conditions.

Hint: Use problem 2 and $\lim_{x \rightarrow \infty} \sin x/x = 0$.

(b) Verify that another particular solution,

$$G_A(t - t') = \frac{1}{\omega}\theta(t' - t) \sin \omega(t - t'), \quad (13)$$

which is referred to as the advanced Green's function, satisfies the Green function differential equation and the continuity conditions.

Hint: Use problem 2 and $\lim_{x \rightarrow \infty} \sin x/x = 0$.

(c) Show that the difference of the two particular solutions above,

$$G_R(t - t') - G_A(t - t'), \quad (14)$$

satisfies the homogeneous differential equations

$$-\left(\frac{d^2}{dt^2} + \omega^2\right)G_0(t, t') = 0. \quad (15)$$