Homework No. 10 (Fall 2019)

PHYS 500A: Mathematical Methods

Due date: Thursday, 2019 Nov 21, 4.00pm

1. (**20 points.**) Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{1}$$

to prove the recurrence relations

$$2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{2a}$$

$$2\frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{2b}$$

2. (**10 points.**) Using the recurrence relations of Eq. (2), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \tag{3}$$

and from this derive the differential equation satisfied by $J_m(t)$.

3. (**20 points.**) Using the recurrence relations,

$$2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{4a}$$

$$2\frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t), \tag{4b}$$

satisfied by the Bessel functions, derive the 'ladder' operations satisfied by the Bessel functions,

$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t), \tag{5}$$

$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t). \tag{6}$$

In quantum mechanics a ladder operator is a raising or lowering operator that transforms eigenfunctions by increasing or decreasing the eigenvalue.