# Homework No. 11 (Fall 2019)

## PHYS 500A: Mathematical Methods

Due date: Thursday, 2019 Dec 5, 4.00pm

1. (**20 points.**) Integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer $m$ and $0 \leq t < \infty$ are

$$K_m(t) = \int_0^\infty d\theta \, \cosh m\theta \, e^{-t\cosh\theta}, \tag{1a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \, e^{t\cos\phi}. \tag{1b}$$

   (a) Using Mathematica (or your favourite graphing tool) plot $K_0(t), K_1(t), K_2(t)$ and $I_0(t), I_1(t), I_2(t)$ on the same plot. (Please do not submit hand sketched plots.)

   (b) Refer Chapter 10 of Digital Library of Mathematical Functions at

   https://dlmf.nist.gov/10

   for a comprehensive resource.

2. (**20 points.**) Show that the integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer $m$ and $0 \leq t < \infty$,

$$K_m(t) = \int_0^\infty d\theta \, \cosh m\theta \, e^{-t\cosh\theta}, \tag{2a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \, e^{t\cos\phi}. \tag{2b}$$

   satisfies the differential equation for modified Bessel functions,

$$\left[ -\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1 \right] \left\{ \begin{array}{c} I_m(t) \\ K_m(t) \end{array} \right\} = 0. \tag{3}$$

   Hint: Integrate by parts, after identifying

$$\left( t\cosh\theta - t^2 \sinh^2\theta \right) e^{-t\cosh\theta} = -\frac{d^2}{d\theta^2} e^{-t\cosh\theta}, \tag{4a}$$

$$\left( t\cos\phi - t^2 \sin^2\phi \right) e^{t\cos\phi} = -\frac{d^2}{d\phi^2} e^{t\cos\phi}. \tag{4b}$$

3. (**20 points.**) The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[ -\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1 \right] \left\{ \begin{array}{c} I_m(t) \\ K_m(t) \end{array} \right\} = 0. \tag{5}$$

Derive the identity, for the Wronskian, (upto a constant $C$)

$$I_m(t)K'_m(t) - K_m(t)I'_m(t) = -\frac{C}{t}, \tag{6}$$

where

$$I'_m(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K'_m(t) \equiv \frac{d}{dt}K_m(t). \tag{7}$$

Further, determine the value of the constant $C$ on the right hand side of Eq. (6) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{8}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{9}$$