Resource material on

Vector calculus in cylindrical polar coordinates

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1. (**10 points.**) In cylindrical polar coordinates a point in space is coordinatized by the intersection of family of right circular cylinders, half-planes, and planes, given by

$$\rho = \sqrt{x^2 + y^2}, \tag{1a}$$
$$\phi = \tan^{-1} \frac{y}{x}, \tag{1b}$$
$$z = z, \tag{1c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla}\rho = \hat{\boldsymbol{\rho}}, \qquad \hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$
$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$
$$\boldsymbol{\nabla}z = \hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) The action of the gradient operator in cylindrical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{3}$$

will involve the derivatives of the unit vectors in cylindrical polar coordinates. Evaluate the following

$$\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\mathbf{z}} = 0, \tag{4a}$$
$$\frac{\partial}{\partial\phi}\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}, \qquad \frac{\partial}{\partial\phi}\hat{\mathbf{z}} = 0, \tag{4b}$$
$$\frac{\partial}{\partial z}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial z}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial z}\hat{\mathbf{z}} = 0. \tag{4c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}, \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{z}}, \tag{5a}$$
$$\boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\rho}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\phi}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\mathbf{z}}), \tag{5b}$$
$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{z}}}{\rho}\right). \tag{5c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla} \times \hat{\boldsymbol{\rho}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\phi}}, \quad \boldsymbol{\nabla} \times \hat{\mathbf{z}}, \tag{6a}$$

$$\boldsymbol{\nabla} \times (\rho^2 \hat{\boldsymbol{\rho}}), \quad \boldsymbol{\nabla} \times (\rho^2 \hat{\boldsymbol{\phi}}), \quad \boldsymbol{\nabla} \times (\rho^2 \hat{\mathbf{z}}), \tag{6b}$$

$$\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\mathbf{z}}}{\rho} \right). \tag{6c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.

5. (**20 points.**) For studying a phenomenon on a plane is it convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}} \frac{\partial}{\partial z}, \tag{7}$$

$$\boldsymbol{\nabla}_\rho = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi}. \tag{8}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right) = 2\pi \delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right) = 0, \tag{9a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right) = \hat{\mathbf{z}}\, 2\pi \delta^{(2)}(\boldsymbol{\rho}). \tag{9b}$$

6. (**30 points.**) The scale factors for cylindrical polar coordinates are

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1. \tag{10}$$

The differential statement in rectangular coordinates is

$$d\mathbf{r} = dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \tag{11}$$

and the corresponding differential statement in cylindrical polar coordinates is

$$d\mathbf{r} = h_\rho d\rho\, \hat{\boldsymbol{\rho}} + h_\phi d\phi\, \hat{\boldsymbol{\phi}} + h_z dz\, \hat{\mathbf{z}}. \tag{12}$$

The gradient operator in rectangular coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \tag{13}$$

and in cylindrical polar coordinates it is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{1}{h_\rho} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{h_\phi} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{1}{h_z} \frac{\partial}{\partial z}. \tag{14}$$

Let a vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x, y, z) + \hat{\mathbf{j}}E_y(x, y, z) + \hat{\mathbf{k}}E_z(x, y, z) \tag{15}$$

be expressed in cylindrical polar coordinates as

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho(\rho, \phi, z) + \hat{\boldsymbol{\phi}}E_\phi(\rho, \phi, z) + \hat{\mathbf{z}}E_z(\rho, \phi, z). \tag{16}$$

Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{h_\rho h_\phi h_z}\left[\frac{\partial}{\partial \rho}(h_\phi h_z E_\rho) + \frac{\partial}{\partial \phi}(h_z h_\rho E_\phi) + \frac{\partial}{\partial z}(h_\rho h_\phi E_z)\right]. \tag{17}$$

Show that

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{h_\rho h_\phi h_z}\begin{vmatrix} h_\rho\hat{\boldsymbol{\rho}} & h_\phi\hat{\boldsymbol{\phi}} & h_z\hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ h_\rho E_\rho & h_\phi E_\phi & h_z E_z \end{vmatrix}. \tag{18}$$

Show that

$$\nabla^2 = \frac{1}{h_\rho h_\phi h_z}\left[\frac{\partial}{\partial \rho}\frac{h_\phi h_z}{h_\rho}\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi}\frac{h_z h_\rho}{h_\phi}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}\frac{h_\rho h_\phi}{h_z}\frac{\partial}{\partial z}\right]. \tag{19}$$