Resource material on

Vector calculus in spherical polar coordinates

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1. (**10 points.**) In spherical polar coordinates a point is coordinated by the intersection of family of spheres, cones, and half-planes, given by

$$r = \sqrt{x^2+y^2+z^2}, \tag{1a}$$

$$\theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}, \tag{1b}$$

$$\phi = \tan^{-1}\frac{y}{x}, \tag{1c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2a}$$

$$\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\theta}}\,\frac{1}{r}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2b}$$

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) Using the gradient operator in spherical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{3}$$

evaluate the following

$$\frac{\partial}{\partial r}\hat{\mathbf{r}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\theta}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\phi}} = 0, \tag{4a}$$

$$\frac{\partial}{\partial\theta}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}, \qquad \frac{\partial}{\partial\theta}\hat{\boldsymbol{\phi}} = 0, \tag{4b}$$

$$\frac{\partial}{\partial\phi}\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}} = -(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}). \tag{4c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{r}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}, \tag{5a}$$

$$\boldsymbol{\nabla}\cdot(r^2\hat{\mathbf{r}}), \quad \boldsymbol{\nabla}\cdot(r^2\hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla}\cdot(r^2\hat{\boldsymbol{\phi}}), \tag{5b}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\theta}}}{r}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{r}\right). \tag{5c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla} \times \hat{\mathbf{r}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\phi}}, \tag{6a}$$

$$\boldsymbol{\nabla} \times (r^2\hat{\mathbf{r}}), \quad \boldsymbol{\nabla} \times (r^2\hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla} \times (r^2\hat{\boldsymbol{\phi}}), \tag{6b}$$

$$\boldsymbol{\nabla} \times \left(\frac{\hat{\mathbf{r}}}{r}\right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\theta}}}{r}\right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\phi}}}{r}\right). \tag{6c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.

5. (**30 points.**) The scale factors for spherical polar coordinates are

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta. \tag{7}$$

The differential statement in rectangular coordinates is

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}} \tag{8}$$

and the corresponding differential statement in spherical polar coordinates is

$$d\mathbf{r} = h_r dr\,\hat{\mathbf{r}} + h_\theta d\theta\,\hat{\boldsymbol{\theta}} + h_\phi d\phi\,\hat{\boldsymbol{\phi}}. \tag{9}$$

The gradient operator in rectangular coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{10}$$

and in spherical polar coordinates it is

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{1}{h_r}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{h_\theta}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{h_\phi}\frac{\partial}{\partial\phi}. \tag{11}$$

Let a vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x,y,z) + \hat{\mathbf{j}}E_y(x,y,z) + \hat{\mathbf{k}}E_z(x,y,z) \tag{12}$$

be expressed in spherical polar coordinates as

$$\mathbf{E} = \hat{\mathbf{r}}E_r(r,\theta,\phi) + \hat{\boldsymbol{\theta}}E_\theta(r,\theta,\phi) + \hat{\boldsymbol{\phi}}E_\phi(r,\theta,\phi). \tag{13}$$

Show that

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial}{\partial r}(h_\theta h_\phi E_r) + \frac{\partial}{\partial\theta}(h_\phi h_r E_\theta) + \frac{\partial}{\partial\phi}(h_r h_\theta E_\phi)\right]. \tag{14}$$

Show that

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\hat{\mathbf{r}} & h_\theta\hat{\boldsymbol{\theta}} & h_\phi\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ h_r E_r & h_\theta E_\theta & h_\phi E_\phi \end{vmatrix}. \tag{15}$$

Show that

$$\nabla^2 = \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial}{\partial r}\frac{h_\theta h_\phi}{h_r}\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\frac{h_\phi h_r}{h_\theta}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{h_r h_\theta}{h_\phi}\frac{\partial}{\partial\phi}\right]. \tag{16}$$