Resource material on

δ-function

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1. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \tag{1}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{2}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{3}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

2. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2} \frac{\epsilon}{(x^2 + \epsilon)^{\frac{3}{2}}}. \tag{4}$$

Show that

$$\delta(x) = \begin{cases} \to \frac{1}{\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{x^2} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{5}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{6}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

3. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}. \tag{7}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{8}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{9}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

4. (**10 points.**) An (idealized) infinitely long wire, (on the z-axis with infinitesimally small cross sectional area,) carrying a current I can be mathematically represented by the current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}}\, I\, \delta(x)\delta(y). \tag{10}$$

A similar idealized wire forms a circular loop and is placed on the xy-plane with the center of the circular loop at the origin. Write down the current density of the circular loop carrying current I.

5. (**10 points.**) A uniformly charged spherical shell of radius a and total charge Q is described by charge density

$$\rho(\mathbf{x}) = \frac{Q}{4\pi a^2}\delta(r-a). \tag{11}$$

Verify that the volume integral of ρ equals Q.

6. (**10 points.**) A uniformly charged infinitely thin disc of radius R and total charge Q is placed on the x-y plane such that the normal vector is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ-function(s).

7. (**10 points.**) Write down the charge density for the following configurations: Point charge, line charge, surface charge, uniformly charged disc, uniformly charged ring, uniformly charged shell, uniformly charged spherical ball.