Final Exam (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Date: 2019 Dec 13

1. (**100 points.**) Consider a point charge q placed on the axis of a perfectly conducting circular cylinder of radius a.

   (a) The relevant Maxwell equation is

   $$\boldsymbol{\nabla}\cdot\varepsilon(\rho)\boldsymbol{\nabla}\phi(\mathbf{r}) = \rho(\mathbf{r}) \tag{1}$$

   with dielectric function

   $$\varepsilon(\rho) = \begin{cases} \varepsilon_0, & \rho < a, \\ \varepsilon_1 \to \infty, & a < \rho, \end{cases} \tag{2}$$

   and charge q at origin

   $$\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}). \tag{3}$$

   The associated Green's function equation is

   $$\boldsymbol{\nabla}\cdot\varepsilon_0\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{4}$$

   with solution

   $$G(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty}\frac{dk_z}{2\pi}e^{ik_z(z-z')}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\phi-\phi')}g_m(\rho,\rho';k_z), \tag{5}$$

   where

   $$g_m(\rho,\rho';k_z) = \frac{1}{\varepsilon_0}\left[I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{K_m(ka)}{I_m(ka)}I_m(k_z\rho_<)I_m(k_z\rho_>)\right]. \tag{6}$$

   (b) Using the connection between the electric potential and the Green function,

   $$\phi(\mathbf{r}) = q\,G(\mathbf{r},\mathbf{r}_0), \tag{7}$$

   where $\mathbf{r}_0 = \mathbf{0}$ is the position of the position of the charge q that is chosen to be at the origin without any loss in generality, determine the electric potential to be

   $$\phi(\mathbf{r}) = \frac{q}{2\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikz}\left[K_0(k\rho) - \frac{K_0(ka)}{I_0(ka)}I_0(k\rho)\right], \tag{8}$$

   where $\mathbf{r} = (\rho,\phi,z)$ is the observation point.

(c) Verify that the potential satisfies the boundary condition

$$\phi(\mathbf{a}) = 0 \tag{9}$$

on the inner surface of the conducting cylinder.

(d) Using the relation $\mathbf{E} = -\boldsymbol{\nabla}\phi$ evaluate the electric field on the inner surface of the conductor to be

$$\mathbf{E}(\mathbf{a}) = \hat{\boldsymbol{\rho}} \frac{q}{2\pi\varepsilon_0} \frac{1}{a} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikz}}{I_0(ka)}. \tag{10}$$

Observe that the electric field is normal to the inner surface of the cylinder. Use the Wronskian.

(e) Using Gauss's law argue that the induced charge on the surface of a conductor is given using

$$\sigma(\phi, z) = \varepsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}\Big|_{\text{surface}}, \tag{11}$$

where $\hat{\mathbf{n}}$ is normal to the surface of conductor. Thus, determine the induced charge density on the inner surface of the cylinder to be

$$\sigma(\phi, z) = -\frac{q}{4\pi^2 a^2} \int_{-\infty}^{\infty} dt \frac{e^{it\frac{z}{a}}}{I_0(t)}. \tag{12}$$

(f) By integrating over the surface of the cylinder determine the total induced charge on the cylinder. Thus, find out if its magnitude is less than, equal to, or greater than, the charge q.