

Homework No. 03 (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Due date: Friday, 2019 Sep 13, 4.00pm

1. **(50 points.)** The Maxwell equations, in SI units, are

$$\nabla \cdot \mathbf{D} = \rho, \quad (1a)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1b)$$

$$-\nabla \times \mathbf{E} - \frac{\partial}{\partial t} \mathbf{B} = 0, \quad (1c)$$

$$\nabla \times \mathbf{H} - \frac{\partial}{\partial t} \mathbf{D} = \mathbf{J}, \quad (1d)$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad (2a)$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}. \quad (2b)$$

The Lorentz force, in SI units, is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \quad (3)$$

We have

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \quad (4)$$

The above quantities will be addressed with subscripts SI in the following. The corresponding quantities in Gaussian (G) units and Heaviside-Lorentz (HL) units are obtained using the conversions

$$\sqrt{\frac{\varepsilon_0}{4\pi}} \mathbf{D}_G = \mathbf{D}_{SI} = \sqrt{\varepsilon_0} \mathbf{D}_{HL}, \quad \sqrt{4\pi\varepsilon_0} \rho_G = \rho_{SI} = \sqrt{\varepsilon_0} \rho_{HL}, \quad (5a)$$

$$\frac{1}{\sqrt{4\pi\varepsilon_0}} \mathbf{E}_G = \mathbf{E}_{SI} = \frac{1}{\sqrt{\varepsilon_0}} \mathbf{E}_{HL}, \quad \sqrt{4\pi\varepsilon_0} \mathbf{P}_G = \mathbf{P}_{SI} = \sqrt{\varepsilon_0} \mathbf{P}_{HL}, \quad (5b)$$

$$\frac{1}{\sqrt{4\pi\mu_0}} \mathbf{H}_G = \mathbf{H}_{SI} = \frac{1}{\sqrt{\mu_0}} \mathbf{H}_{HL}, \quad \sqrt{4\pi\varepsilon_0} \mathbf{J}_G = \mathbf{J}_{SI} = \sqrt{\varepsilon_0} \mathbf{J}_{HL}, \quad (5c)$$

$$\sqrt{\frac{\mu_0}{4\pi}} \mathbf{B}_G = \mathbf{B}_{SI} = \sqrt{\mu_0} \mathbf{B}_{HL}, \quad \sqrt{\frac{4\pi}{\mu_0}} \mathbf{M}_G = \mathbf{M}_{SI} = \frac{1}{\sqrt{\mu_0}} \mathbf{M}_{HL}. \quad (5d)$$

Note that the Heaviside-Lorentz units are obtained from Gaussian units by dropping the 4π 's, which is called rationalization in this context.

- (a) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Gaussian units. The Maxwell equations, in Gaussian units, are

$$\nabla \cdot \mathbf{D}_G = 4\pi\rho_G, \quad (6a)$$

$$\nabla \cdot \mathbf{B}_G = 0, \quad (6b)$$

$$-\nabla \times \mathbf{E}_G - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_G = 0, \quad (6c)$$

$$\nabla \times \mathbf{H}_G - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{D}_G = \frac{4\pi}{c} \mathbf{J}_G, \quad (6d)$$

where

$$\mathbf{D}_G = \mathbf{E}_G + 4\pi\mathbf{P}_G, \quad (7a)$$

$$\mathbf{H}_G = \mathbf{B}_G - 4\pi\mathbf{M}_G. \quad (7b)$$

The Lorentz force, in Gaussian units, is

$$\mathbf{F} = q_G \mathbf{E}_G + q_G \frac{\mathbf{v}}{c} \times \mathbf{B}_G. \quad (8)$$

Here charge q_G has the same conversion as charge density ρ_G .

- (b) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Lorentz-Heaviside units. The Maxwell equations, in Heaviside-Lorentz units, are

$$\nabla \cdot \mathbf{D}_{HL} = \rho_{HL}, \quad (9a)$$

$$\nabla \cdot \mathbf{B}_{HL} = 0, \quad (9b)$$

$$-\nabla \times \mathbf{E}_{HL} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_{HL} = 0, \quad (9c)$$

$$\nabla \times \mathbf{H}_{HL} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{D}_{HL} = \frac{1}{c} \mathbf{J}_{HL}, \quad (9d)$$

where

$$\mathbf{D}_{HL} = \mathbf{E}_{HL} + \mathbf{P}_{HL}, \quad (10a)$$

$$\mathbf{H}_{HL} = \mathbf{B}_{HL} - \mathbf{M}_{HL}. \quad (10b)$$

The Lorentz force, in Heaviside-Lorentz units, is

$$\mathbf{F} = q_{HL} \mathbf{E}_{HL} + q_{HL} \frac{\mathbf{v}}{c} \times \mathbf{B}_{HL}. \quad (11)$$

Here charge q_{HL} has the same conversion as charge density ρ_{HL} .

2. **(20 points.)** In Gaussian units the power radiated by an accelerated charged particle of charge e is given by the Larmor formula,

$$P = \frac{2e^2}{3c^3} a^2, \quad (12)$$

where a is the acceleration of the charged particle. Write down the Larmor formula in SI units, and in Lorentz-Heaviside units.

3. **(30 points.)** The fine-structure constant, in Gaussian units,

$$\alpha = \frac{e^2}{\hbar c}, \quad (13)$$

is the parameter that characterizes the strength of the electromagnetic interaction.

- (a) Write down the corresponding expression for fine-structure constant in SI units, and in Lorentz-Heaviside units.
 - (b) Verify that the fine-structure constant is a dimensionless quantity. Show that the numerical value of the fine-structure constant is independent of the system of units.
 - (c) Evaluate the numerical value for the reciprocal of the fine-structure constant, α^{-1} . (A periodic table based on quantum electrodynamics breaks down for atomic numbers greater than α^{-1} .)
4. **(30 points.)** The relation between charge density and current density,

$$\frac{\partial}{\partial t} \rho(\mathbf{r}, t) + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0, \quad (14)$$

is the general statement of the conservation of charge.

- (a) Derive the statement of conservation of charge in Eq. (14) from the Maxwell equations.

Hint: Take time derivative of Gauss's law and divergence of Ampere's law.

- (b) For an arbitrarily moving point particle with charge q , the charge and current densities are

$$\rho(\mathbf{r}, t) = q\delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t)) \quad (15)$$

and

$$\mathbf{j}(\mathbf{r}, t) = q\mathbf{v}_a(t)\delta^{(3)}(\mathbf{r} - \mathbf{r}_a(t)), \quad (16)$$

where $\mathbf{r}_a(t)$ is the position vector and

$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \quad (17)$$

is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (14) for a point particle.