Homework No. 04 (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2019 Sep 23, 4.00pm

1. (**50 points.**) Consider a uniformly charged spherical ball of radius R with total charge q.

   (a) Using Gauss's law show that the electric field is given by

   $$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{R^3}, & r < R, \\ \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{r^3}, & R < r. \end{cases} \tag{1}$$

   The magnetic field $\mathbf{B} = 0$ everywhere.

   (b) Starting from the equation for conservation of electromagnetic linear momentum we have

   $$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0. \tag{2}$$

   Show that $\mathbf{G} = 0$. Thus, infer

   $$\mathbf{f} \cdot \hat{\mathbf{r}} = -(\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \hat{\mathbf{r}}. \tag{3}$$

   (c) Consider a spherical volume V of radius r with the charge at the center. Note that $\mathbf{F} = \int d^3r\, \mathbf{f}$ will be zero due to spherical symmetry. To determine the electromagnetic stress (or the pressure, force per unit area,) on the sphere due to electrostatic repulsion between the constituent charges we define $F = \int_V d^3r\, (\mathbf{f} \cdot \hat{\mathbf{r}})$, which is the total sum of radial component of forces exerted on all the charges inside volume V by the electric and magnetic fields. The interpretation of this quantity as the force contributing to the pressure on the charge is brought out after we use divergence theorem to learn

   $$F = -\oint_V d\mathbf{a} \cdot \mathbf{T} \cdot \hat{\mathbf{r}} = -\oint_V da\, \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}}, \tag{4}$$

   where we used $d\mathbf{a} = da\, \hat{\mathbf{r}}$. That is, F is the radial force on the charge due to the flux of electromagnetic momentum across the surface enclosing volume V. Note that $F \neq \mathbf{F} \cdot \hat{\mathbf{r}}$, because $\mathbf{F} = 0$.

   (d) Evaluate

   $$\hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}} = \begin{cases} -\dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{1}{r^4}, & r < R, \\ -\dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^2}{R^6}, & R < r. \end{cases} \tag{5}$$

(e) Thus, calculate the radial force F on the surface of the charge to be

$$F = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}. \tag{6}$$

Then, calculate the electromagnetic stress, F/area, on the surface of the charged sphere. For arbitrary volume V of radius r show that

$$F = \begin{cases} \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{r^2}, & r < R, \\ \frac{1}{2}\frac{q^2}{4\pi\varepsilon_0}\frac{r^4}{R^6}, & R < r. \end{cases} \tag{7}$$

What is the interpretation? What can you conclude about the electromagnetic stress at a point inside the charge from the above relation?

(f) Next, consider the volume V to be a spherical shell of inner radius b and outer radius b', such that $a < b < b'$. Since there is no charge enclosed in the shell we expect $F = 0$. Evaluate F using Eq. (4). Interpret your result.

2. (**20 points.**) Consider a charge distribution consisting of two point charges with charges equal in magnitude and opposite in sign. The positive charge $+q$ is fixed at position $\mathbf{a}$ on the z axis, and the negative charge $-q$ is fixed at position $-\mathbf{a}$ on the z axis, such that the two charges have a dipole moment $\mathbf{p} = 2q\mathbf{a}$. The electric field for the configuration is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} - \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^3} \tag{8}$$

and the magnetic field $\mathbf{B} = 0$ everywhere.

(a) Starting from the equation for conservation of electromagnetic linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0, \tag{9}$$

infer that total force along the direction of $\hat{\mathbf{z}}$ on the charges inside volume V is given by

$$F_z = \int_V d^3r\,\mathbf{f}\cdot\hat{\mathbf{z}} = -\oint_V da\,\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{z}}. \tag{10}$$

(b) Let us choose the volume V to be a sphere of radius δ centered at the negative charge. Thus, the points on the surface of this sphere satisfy

$$\mathbf{r} = -\mathbf{a} + \boldsymbol{\delta}. \tag{11}$$

Let $\delta \ll a$. Verify that

$$F_z = -\oint_V da\,\hat{\boldsymbol{\delta}}\cdot\left[\mathbf{1}\frac{1}{2}\varepsilon_0 E^2 - \varepsilon_0\mathbf{E}\mathbf{E}\right]\cdot\hat{\mathbf{z}}. \tag{12}$$

(c) Let $\hat{\boldsymbol{\delta}} \cdot \hat{\mathbf{z}} = \cos\theta$. Show that

$$F_z = \lim_{\delta\to 0} \frac{1}{2} \frac{q^2}{4\pi\varepsilon_0} \int_0^\pi \sin\theta d\theta \left[\frac{\cos\theta}{2\delta^2} + \frac{1}{(2a)^2} \right] = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{(2a)^2}, \tag{13}$$

which is the Coulomb force on the negative charge.
Hints: The following intermediate steps should help in the above evaluation. Show that on the surface of volume V

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{(-2\mathbf{a} + \boldsymbol{\delta})}{|-2\mathbf{a} + \boldsymbol{\delta}|^3} - \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{\delta}}{\delta^3} = -\frac{q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\delta}}}{\delta^2} - \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{z}}}{(2a)^2} + \mathcal{O}(\delta). \tag{14}$$

Further, verify that

$$\hat{\boldsymbol{\delta}} \cdot \mathbf{E} = -\frac{q}{4\pi\varepsilon_0} \frac{1}{\delta^2} - \frac{q}{4\pi\varepsilon_0} \frac{\cos\theta}{(2a)^2} + \mathcal{O}(\delta), \tag{15}$$

$$\mathbf{E} \cdot \hat{\mathbf{z}} = -\frac{q}{4\pi\varepsilon_0} \frac{\cos\theta}{\delta^2} - \frac{q}{4\pi\varepsilon_0} \frac{1}{(2a)^2} + \mathcal{O}(\delta). \tag{16}$$

(d) The first equality in Eq. (13) is divergent in the limit $\delta \to 0$. However it goes to zero if the θ integral is completed before taking the limit. Interpret this argument.