

Homework No. 05 (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Due date: Friday, 2019 Oct 11, 4.00pm

1. **(20 points.)** Show that the effective charge density, ρ_{eff} , and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\nabla \cdot \mathbf{P}, \quad (1)$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t} \mathbf{P} + \nabla \times \mathbf{M}, \quad (2)$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t} \rho_{\text{eff}} + \nabla \cdot \mathbf{j}_{\text{eff}} = 0. \quad (3)$$

2. **(60 points.)** Consider a solid cylinder of radius R and infinite length with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0 \theta(R - \rho), \quad (4)$$

where $\rho^2 = x^2 + y^2$ and $\mathbf{P}_0$ is perpendicular to the axis of the cylinder. We shall find the electric potential and the electric field outside the cylinder.

- (a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\nabla \cdot \mathbf{P} = \mathbf{P}_0 \cdot \hat{\boldsymbol{\rho}} \delta(\rho - R). \quad (5)$$

- (b) Beginning from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (6)$$

after integrating by parts, and writing

$$\phi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \mathbf{P}_0 \cdot \nabla \int d^3r' \frac{\theta(R - \rho')}{|\mathbf{r} - \mathbf{r}'|}, \quad (7)$$

show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \theta(R - \rho') \frac{\mathbf{P}_0 \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (8)$$

- (c) Evaluate the integrals, z' , ϕ' , and ρ' , to show that the electric potential outside the cylinder is given by

$$\phi(\mathbf{r}) = \frac{2\pi R^2}{4\pi\epsilon_0} \frac{\mathbf{P}_0 \cdot \boldsymbol{\rho}}{\rho^2}. \quad (9)$$

Hints:

i. In cylindrical coordinates we have

$$\boldsymbol{\rho} = \rho \cos \phi \hat{\mathbf{i}} + \rho \sin \phi \hat{\mathbf{j}}, \quad \mathbf{r} = \boldsymbol{\rho} + z \hat{\mathbf{k}}, \quad (10a)$$

$$\boldsymbol{\rho}' = \rho' \cos \phi' \hat{\mathbf{i}} + \rho' \sin \phi' \hat{\mathbf{j}} \quad \mathbf{r}' = \boldsymbol{\rho}' + z' \hat{\mathbf{k}}, \quad (10b)$$

$$\mathbf{P}_0 = P_0 \cos \alpha \hat{\mathbf{i}} + P_0 \sin \alpha \hat{\mathbf{j}}. \quad (10c)$$

Thus,

$$|\mathbf{r} - \mathbf{r}'|^2 = (z - z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2 \quad (11)$$

and

$$\mathbf{P}_0 \cdot (\mathbf{r} - \mathbf{r}') = \mathbf{P}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}'). \quad (12)$$

ii. Complete the z' integral using

$$\int_{-\infty}^{\infty} \frac{dz}{(z^2 + a^2)^{\frac{3}{2}}} = \frac{2}{a^2} \quad (13)$$

to obtain the result

$$\phi(\mathbf{r}) = \frac{2}{4\pi\epsilon_0} \int d^2\rho' \theta(R - \rho') \frac{\mathbf{P}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2}. \quad (14)$$

where $d^2\rho' = \rho' d\rho' d\phi'$.

iii. Choose $\phi = 0$. Then, complete the ϕ' integrals using

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{(1 \pm a \cos \phi)} = \frac{1}{\sqrt{1 - a^2}}, \quad |a| < 1, \quad (15)$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{\cos \phi}{(1 \pm a \cos \phi)} = \pm \frac{1}{a} \mp \frac{1}{a\sqrt{1 - a^2}}, \quad |a| < 1, \quad (16)$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{\sin \phi}{(1 \pm a \cos \phi)} = 0, \quad |a| < 1. \quad (17)$$

iv. Collect (and complete cancellations) before completing the ρ' integral. The divergence associated to $\rho \rightarrow R$ cancels for $\rho' < R < \rho$.

(d) Evaluate the gradient of the electric potential to show that the electric field outside the cylinder is given by

$$\mathbf{E}(\mathbf{r}) = \frac{2\pi R^2}{4\pi\epsilon_0} \frac{1}{\rho^2} \left[2(\mathbf{P}_0 \cdot \hat{\boldsymbol{\rho}}) \hat{\boldsymbol{\rho}} - \mathbf{P}_0 \right]. \quad (18)$$