Homework No. 09 (Fall 2019)

PHYS 520A: Electromagnetic Theory I

Due date: Friday, 2019 Dec 6, 4.00pm

1. (**20 points.**) The free Green's function satisfies

$$\varepsilon_0 \nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{1}$$

The free Green's function is the electric potential at $\mathbf{r}$ due to a point charge of unit magnitude at $\mathbf{r}'$,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}')|}. \tag{2}$$

To derive a representation for the free Green's function suitable for cylindrically symmetric configurations, we require translational symmetry in the z and rotational symmetry in ϕ, in terms cylindrical coordinates (ρ, ϕ, z),

$$G(\mathbf{r}, \mathbf{r}') = G(\rho, \rho', \phi - \phi', z - z'). \tag{3}$$

In cylindrical coordinates we have

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}') = \delta(\rho - \rho') \frac{\delta(\phi - \phi')}{\rho} \delta(z - z'). \tag{4}$$

Use Fourier transformations to write

$$G(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} g_m(\rho, \rho'; k_z) \tag{5}$$

and

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}') = \frac{\delta(\rho - \rho')}{\rho} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')}. \tag{6}$$

Then, show that the reduced free Green's function satisfies the differential equation

$$\left[-\frac{1}{\rho} \frac{\partial}{\partial\rho} \rho \frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2 \right] \varepsilon_0 g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \tag{7}$$

2. (**40 points.**) The cylindrical free Green's function satisfies

$$\Big[-\frac{1}{\rho} \frac{\partial}{\partial\rho} \rho \frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2 \Big] \varepsilon_0 g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \tag{8}$$

(a) Integrate Eq. (8) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{9a}$$

$$\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z)\Bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -\frac{1}{\varepsilon_0}. \tag{9b}$$

(b) Let us further require that

$$g_m(0, \rho'; k_z) \text{ is finite,} \tag{10a}$$

$$g_m(\infty, \rho'; k_z) = 0. \tag{10b}$$

(c) Recall the Wronskian

$$I_m(t) K_m'(t) - I_m'(t) K_m(t) = -\frac{1}{t}. \tag{11}$$

(d) Derive the solution

$$g_m(\rho, \rho') = \frac{1}{\varepsilon_0} I_m(k_z \rho_<) K_m(k_z \rho_>). \tag{12}$$

3. (**80 points.**) The cylindrical Green's function satisfies

$$\Big[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \varepsilon(\rho) \rho \frac{\partial}{\partial \rho} + \varepsilon(\rho) \frac{m^2}{\rho^2} + \varepsilon(\rho) k_z^2 \Big] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \tag{13}$$

Consider a dielectric cylinder sorrounded by a perfectly conducting cylinder described by

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2 & \text{for} \quad \rho < a, \\ \infty & \text{for} \quad a < \rho. \end{cases} \tag{14}$$

(a) Integrate Eq. (13) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{15a}$$

$$\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z)\Bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -\frac{1}{\varepsilon_2}. \tag{15b}$$

(b) Using the property of a perfectly conducting cylinder we require

$$g_m(0, \rho'; k_z) \text{ is finite,} \tag{16a}$$

$$g_m(a, \rho'; k_z) = 0. \tag{16b}$$

(c) Recall the Wronskian

$$I_m(t) K_m'(t) - I_m'(t) K_m(t) = -\frac{1}{t}. \tag{17}$$

(d) Derive the solution

$$g_m(\rho, \rho') = \frac{1}{\varepsilon_2} I_m(k_z \rho_<) K_m(k_z \rho_>) - \frac{1}{\varepsilon_2} \frac{K_m(k_z a)}{I_m(k_z a)} I_m(k_z \rho) I_m(k_z \rho'). \tag{18}$$