Midterm Exam No. 03 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

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Instruction: You are allowed to refer to any resource of your choice. Choose to be honest and fair to your peers. After completing the exam, scan your work and submit it as a single PDF file to `shajesh@siu.edu`. This exam is due on the official date of the exam.

Note: Standard identities will be provided to a student when requested.

1. (**20 points.**) The Pauli matrices are traceless Hermitian matrices that satisfy
$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k, \tag{1}$$
where i, j, are either $1, 2$, or 3. Evaluate the commutation relations
$$\big[\sigma_i, \sigma_j\big]. \tag{2}$$
Commutation relation between two matrices A and B is defined as
$$\big[A, B\big] = AB - BA. \tag{3}$$

2. (**30 points.**) Evaluate the inverse
$$(\sigma_3)^{-1}. \tag{4}$$
Is σ_3 unitary? Then, evaluate
$$(\sigma_1\sigma_2)^{-1}. \tag{5}$$
Is $(\sigma_1\sigma_2)$ unitary?

3. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation
$$\left[\frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2\right] x(t) = 0, \tag{6}$$
where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Plot $x(t)$ as a function of t in the graph in Figure 1 where $x_0\, e^{-\omega_0 t}$ is already plotted for reference. For what t is the solution $x(t)$ a maximum?

4. (**20 points.**) Plot
$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t\right] \tag{7}$$
versus t for $x_0 = 0$ for $\omega_0 > \gamma$ and $\omega_0 < \gamma$. Measure t in units of $1/\omega_0$. Evaluate the maximum value attained by $x(t)$ and mark it on the plot. A neat qualitatively correct plot drawn by hand is sufficient A plot drawn using a software is also fine.

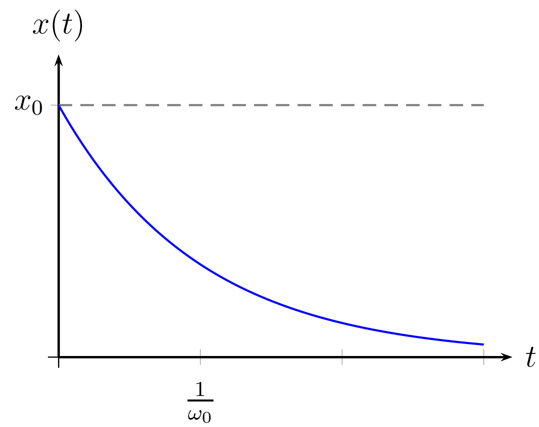

Figure 1: Critically damped harmonic oscillator.

5. (**20 points.**) Verify that

$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \tag{8}$$

where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that

$$\frac{d^2}{dz^2}|z| = 2\,\delta(z). \tag{9}$$

Also, argue that, for a well defined function $f(z)$, the replacement

$$f(z)\delta(z) = f(0)\delta(z) \tag{10}$$

is justified. Using Eq. (8), Eq. (9), and Eq. (10), verify (by substituting the solution into the differential equation) that

$$g(z) = \frac{1}{2k}\, e^{-k|z|} \tag{11}$$

is a particular solution of the differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z) = \delta(z). \tag{12}$$